Complementarity Constraint Qualifications and Simplified B-Stationarity Conditions for Mathematical Programs with Equilibrium Constraints*

Jong-Shi Pang† and Masao Fukushima‡

January 24, 1998

Abstract. With the aid of some novel complementarity constraint qualifications, we derive some simplified primal-dual characterizations of a B-stationary point for a mathematical program with complementarity constraints (MPEC). The approach is based on a locally equivalent piecewise formulation of such a program near a feasible point. The simplified results, which rely heavily on a careful dissection and improved understanding of the tangent cone of the feasible region of the program, bypass the combinatorial characterization that is intrinsic to B-stationarity.

Keywords. Mathematical programs with equilibrium constraints, stationarity conditions.

AMS Subject Classification. 90C30, 90C33

1 Introduction

We consider a mathematical program with equilibrium constraints (an MPEC) formulated as the following constrained optimization problem with complementarity conditions (see [16]):

$$\begin{array}{ll} \text{minimize} & \theta(z) \\ \text{subject to} & g(z) \leq 0, \quad h(z) = 0 \\ \text{and} & 0 \leq G(z) \perp H(z) \geq 0, \end{array} \tag{1}$$

where $\theta : \Re^N \to \Re$, $g : \Re^N \to \Re^m$, $h : \Re^N \to \Re^\ell$, $G : \Re^N \to \Re^M$, and $H : \Re^N \to \Re^M$ are given continuously differentiable functions. This formulation is very broad and provides a useful

*This work was based on research supported partly by a Japan-US collaborative grant awarded to the authors by the U.S. National Science Foundation under grant INT-9417085 and the Japan Society for the Promotion of Science. The first author's research was also supported by the U.S. National Science Foundation under grant CCR-9624018. The second author's research was also supported in part by the Scientific Research Grant-in-Aid from the Ministry of Education, Science, Sports, and Culture of Japan.

†Department of Mathematical Sciences, Whiting School of Engineering, The Johns Hopkins University, Baltimore, Maryland 21218-2682, U.S.A. (jsp@vicp1.mts.jhu.edu)

‡Department of Applied Mathematics and Physics, Graduate School of Engineering, Kyoto University, Kyoto 606-01, Japan (fuku@kuamp.kyoto-u.ac.jp).

framework for the study of all bilevel optimization problems. In the form (1), all variables of the problem (first-level, second-level, and auxiliary) are modeled by the vector z; the equilibrium constraints are expressed by the complementarity system:

$$0 \leq G(z) \perp H(z) \geq 0;$$

all other constraints (including the first-level constraints and all equality constraints that might occur in a mixed complementarity formulation of the second-level problem) are described by

$$g(z) \leq 0, \quad h(z) = 0.$$

Notice that joint constraints between the first-level and second-level variables can be modeled by the above formulation; see [7] for discussion on some technical difficulty associated with MPECs with joint constraints of this kind.

Stationarity (or first-order optimality) conditions for MPECs and/or bilevel optimization problems have been the subject of many recent papers [5, 12, 13, 14, 16, 18, 22, 19, 20, 23]; see also [4] which contains a session that discusses these conditions. Since there are several different approaches for deriving these conditions, various stationarity concepts arise. In the recent paper [16], Scheel and Scholtes have made an excellent attempt to clarify these concepts and elucidate their connections. Among these stationarity conditions, the one introduced in the paper [12] and studied in depth in the monograph [10] appears to provide the strongest first-order optimality conditions for a local minimizer of an MPEC. Scheel and Scholtes coined the term "B-stationarity" (where "B" stands for Bouligand).

In order to formally define a B-stationary point, we recall the notion of a tangent cone. Specifically, for any set $X \subseteq \Re^n$ and a vector $x \in X$, the tangent cone of X at x, denoted $\mathcal{T}(x; X)$, is the set consisting of all vectors dx for which there exist sequences $\{x^\nu\} \subset X$ and $\{\tau_\nu\} \subset \Re_{++}$ such that

$$\lim_{\nu \to \infty} \tau_\nu = 0 \quad \text{and} \quad \lim_{\nu \to \infty} \frac{x^\nu - x}{\tau_\nu} = dx.$$

Let $\mathcal{F}$ denote the feasible region of the MPEC (1) and let z^* be a given vector in $\mathcal{F}$. We say that z^* is *B-stationary* for the MPEC (1) if

$$\nabla\theta(z^*)^T dz \geq 0, \quad \forall\, dz \in \mathcal{T}(z^*; \mathcal{F});$$

or equivalently,

$$\nabla\theta(z^*) \in \mathcal{T}(z^*; \mathcal{F})^*, \tag{2}$$

where the dual cone $\mathcal{C}^*$ of an arbitrary cone $\mathcal{C} \in \Re^N$ is defined by

$$\mathcal{C}^* \equiv \{\, v \in \Re^N : y^T v \geq 0, \, \forall\, y \in \mathcal{C} \,\}.$$

As previous studies have shown (for details see the above cited references, particularly [12, 10]), the practical difficulty of establishing the B-stationarity of a feasible point z^* of an MPEC lies in the complex (in particular, nonconvex) nature of the tangent cone $\mathcal{T}(z^*; \mathcal{F})$. Our goal in this paper is to continue our effort in [12, 10] to obtain some further understanding of this

cone. More specifically, we wish to derive some new sufficient MPEC hypotheses under which the B-stationarity of z^* can be easily described. Our approach lies in demonstrating that under these favorable MPEC hypotheses, the representation of the cone $\mathcal{T}(z^*;\mathcal{F})$ and/or its dual cone becomes much simpler. In turn, these simplified representations enable us to obtain the improved B-stationarity conditions that resemble the well-known Karush-Kuhn-Tucker optimality conditions for a smooth nonlinear program satisfying some standard constraint qualifications, thus somewhat easing the extreme difficulty of a general MPEC. Also, from a computational viewpoint, it is conceivable that the results obtained in this paper will advance the convergence analysis of iterative algorithms for computing B-stationary points of MPECs which to date invariably requires a nondegeneracy assumption [10, 6]. The detailed investigation of such algorithmic aspects will not be persued here.

2 The Piecewise Programming Approach

The approach is based on piecewise programming in which we focus on the finite number of pieces that compose the feasible region of the MPEC near z^*. Since we will work extensively with index sets and their particular subsets, it will be convenient for us to introduce a notation. Specifically, for a given finite index set $\mathcal{J}$, let $\mathcal{P}(\mathcal{J})$ denote the family of pairs $(\mathcal{J}_1, \mathcal{J}_2)$ of partitioning subsets of $\mathcal{J}$, i.e.,

$$\mathcal{P}(\mathcal{J}) \equiv \{\, (\mathcal{J}_1, \mathcal{J}_2) : \mathcal{J}_1 \cup \mathcal{J}_2 = \mathcal{J},\ \mathcal{J}_1 \cap \mathcal{J}_2 = \emptyset \,\}.$$

Associated with the given feasible vector z^*, we define several index sets:

$$\begin{aligned}
\alpha &\equiv \{\, j : G_j(z^*) = 0 < H_j(z^*) \,\} \\
\beta &\equiv \{\, j : G_j(z^*) = 0 = H_j(z^*) \,\} \\
\gamma &\equiv \{\, j : G_j(z^*) > 0 = H_j(z^*) \,\} \\
\mathcal{I} &\equiv \{\, k : g_k(z^*) = 0 \,\}.
\end{aligned}$$

In particular, β is the *degenerate set* at z^*. If β is empty, then z^* is a *nondegenerate* feasible solution of the MPEC. This note focuses on the case where β is nonempty, because this is the source of difficulty for the MPEC. Corresponding to the first three index sets α, β and γ, we define several smooth nonlinear programs (NLPs) associated with z^* (see [10]).

The relaxed NLP, denoted RNLP_*:

$$\begin{aligned}
\text{minimize} \quad & \theta(z) \\
\text{subject to} \quad & g(z) \le 0, \quad h(z) = 0 \\
& 0 = G_\alpha(z), \quad H_\alpha(z) \ge 0 \\
& 0 \le G_\beta(z), \quad H_\beta(z) \ge 0 \\
& 0 \le G_\gamma(z), \quad H_\gamma(z) = 0.
\end{aligned}$$

The (β_1, β_2) restricted NLP, denoted $\mathrm{NLP}_*(\beta_1, \beta_2)$, where $(\beta_1, \beta_2) \in \mathcal{P}(\beta)$:

$$\begin{aligned} \text{minimize} \quad & \theta(z) \\ \text{subject to} \quad & g(z) \leq 0, \quad h(z) = 0 \\ & 0 = G_{\alpha_1}(z), \quad H_{\alpha_1}(z) \geq 0 \\ & 0 \leq G_{\gamma_2}(z), \quad H_{\gamma_2}(z) = 0, \end{aligned}$$

where

$$\alpha_1 \equiv \alpha \cup \beta_1 \quad \text{and} \quad \gamma_2 \equiv \gamma \cup \beta_2.$$

These NLPs are known to provide useful local information for the MPEC (1) near z^*, analytically [10, 16] and algorithmically [9, 11, 15]. They continue to play a central role in the present study.

Let $\mathcal{F}_{\mathrm{R}}$ and $\mathcal{F}_{(\beta_1,\beta_2)}$ denote the feasible regions of the RNLP_* and $\mathrm{NLP}_*(\beta_1, \beta_2)$, respectively. The vector z^* clearly belongs to these regions. Moreover there exists a neighborhood $\mathcal{V} \subseteq \Re^N$ of z^* such that

$$\bigcup_{(\beta_1,\beta_2)\in\mathcal{P}(\beta)} \left(\mathcal{V} \cap \mathcal{F}_{(\beta_1,\beta_2)} \right) = \mathcal{V} \cap \mathcal{F} \subseteq \mathcal{V} \cap \mathcal{F}_{\mathrm{R}}.$$

Since the above union involves only finitely many pairs, it follows that

$$\bigcup_{(\beta_1,\beta_2)\in\mathcal{P}(\beta)} \mathcal{T}(z^*; \mathcal{F}_{(\beta_1,\beta_2)}) = \mathcal{T}(z^*; \mathcal{F}) \subseteq \mathcal{T}(z^*; \mathcal{F}_{\mathrm{R}}). \tag{3}$$

Dualizing the above expression, we obtain

$$\mathcal{T}(z^*; \mathcal{F}_{\mathrm{R}})^* \subseteq \left(\bigcup_{(\beta_1,\beta_2)\in\mathcal{P}(\beta)} \mathcal{T}(z^*; \mathcal{F}_{(\beta_1,\beta_2)}) \right)^*. \tag{4}$$

This paper addresses the following three main questions and studies their consequences:

(I) When does the inclusion in (3) hold as an equality?

(II) When does equality hold in (4)?

(III) What if the RNLP_* is "overrelaxing"? More precisely, if equality fails to hold in (4), are there cases where $\mathcal{T}(z^*; \mathcal{F})^*$ can be represented "constructively"?

First, we obtain a necessary and sufficient condition for question (I). Clearly, if the inclusion in (3) holds as an equality, then equality holds in (4). However, when $\mathcal{T}(z^*; \mathcal{F})$ is a proper subset of $\mathcal{T}(z^*; \mathcal{F}_{\mathrm{R}})$, inclusion (4) may hold either as equality (see Example 1 below) or strictly (see Example 2 below). To gain a better understanding of what makes such a distinction, we consider the second question (II) independently of the first. This leads us to the definition of some new MPEC hypotheses which will be used to study the second and third question. Incidentally, $\mathcal{T}(z^*; \mathcal{F})$ is nonconvex in Example 1; whereas $\mathcal{T}(z^*; \mathcal{F})$ is convex in Example 2.

Example 1. Consider the case where the functions g and h are absent and

$$G(z) \equiv z_1 \quad \text{and} \quad H(z) \equiv z_2, \quad \text{for } z = (z_1, z_2) \in \Re^2.$$

Let $z^* \equiv (0,0)$. Then

$$\alpha = \gamma = \emptyset, \quad \text{and} \quad \beta = \{1\}.$$

We have $\mathcal{P}(\beta) = \{(\{1\},\emptyset),(\emptyset,\{1\})\}$. Thus

$$\begin{aligned} \mathcal{T}(z^*;\mathcal{F}) &= \bigcup_{(\beta_1,\beta_2)\in\mathcal{P}(\beta)} \mathcal{T}(z^*;\mathcal{F}_{(\beta_1,\beta_2)}) \\ &= \{\,(dz_1,dz_2) \in \Re^2_+ : dz_1 = 0 \text{ or } dz_2 = 0\,\}. \end{aligned}$$

However, since

$$\mathcal{T}(z^*;\mathcal{F}_{\mathrm{R}}) = \Re^2_+,$$

the cone $\mathcal{T}(z^*;\mathcal{F})$ is a proper subset of the cone $\mathcal{T}(z^*;\mathcal{F}_{\mathrm{R}})$. Nevertheless, the dual cones of $\mathcal{T}(z^*;\mathcal{F}_{\mathrm{R}})$ and $\mathcal{T}(z^*;\mathcal{F})$ are both equal to $\Re^2_+$.

Example 2. Consider the case where the function h is absent and

$$G(z) \equiv z_1, \quad H(z) \equiv z_1 + z_2, \quad \text{and} \quad g(z) \equiv z_2, \quad \text{for } z = (z_1,z_2) \in \Re^2.$$

Let $z^* \equiv (0,0)$. Then

$$\alpha = \gamma = \emptyset, \quad \text{and} \quad \beta = \{1\}.$$

We have $\mathcal{P}(\beta) = \{(\{1\},\emptyset),(\emptyset,\{1\})\}$ and

$$\begin{aligned} \mathcal{T}(z^*;\mathcal{F}) &= \bigcup_{(\beta_1,\beta_2)\in\mathcal{P}(\beta)} \mathcal{T}(z^*;\mathcal{F}_{(\beta_1,\beta_2)}) \\ &= \{\,(dz_1,dz_2) \in \Re^2 : dz_1 + dz_2 = 0 \text{ and } dz_2 \le 0\,\}, \end{aligned}$$

while

$$\mathcal{T}(z^*;\mathcal{F}_{\mathrm{R}}) = \{\,(dz_1,dz_2) \in \Re^2 : dz_1 + dz_2 \ge 0 \text{ and } dz_2 \le 0\,\}.$$

Thus the (convex) cone $\mathcal{T}(z^*;\mathcal{F})$ is a proper subset of $\mathcal{T}(z^*;\mathcal{F}_{\mathrm{R}})$. Moreover, we have

$$\mathcal{T}(z^*;\mathcal{F}_{\mathrm{R}})^* = \{\,(v_1,v_2) \in \Re^2 : v_1 - v_2 \ge 0 \text{ and } v_1 \ge 0\,\}$$

and

$$\mathcal{T}(z^*;\mathcal{F})^* = \{\,(v_1,v_2) \in \Re^2 : v_1 - v_2 \ge 0\,\},$$

which shows that the dual cone $\mathcal{T}(z^*;\mathcal{F}_{\mathrm{R}})^*$ is also a proper subset of $\mathcal{T}(z^*;\mathcal{F})^*$.

The above examples suggest that, in the degenerate case $\beta \neq \emptyset$, the equality

$$\mathcal{T}(z^*;\mathcal{F}_{\mathrm{R}}) = \mathcal{T}(z^*;\mathcal{F})$$

will hold only under restrictive situations. Nevertheless it seems reasonable to expect the equality

$$\mathcal{T}(z^*;\mathcal{F}_{\mathrm{R}})^* = \mathcal{T}(z^*;\mathcal{F})^*. \tag{5}$$

to hold more frequently. As we see from (2), the equality (5) is practically significant because the dual cone $\mathcal{T}(z^*;\mathcal{F})^*$ provides the complete characterization of a B-stationary point of the MPEC. Equating this dual cone with $\mathcal{T}(z^*;\mathcal{F}_{\rm R})^*$ therefore allows us to obtain a much simplified description of such a point.

Consideration of the above questions is motivated by the fact that the tangent cone $\mathcal{T}(z^*;\mathcal{F})$ is in general very difficult to describe effectively. Our results are aimed at identifying situations in which this cone or its dual can be described in a simple way. Previous studies (see [10] e.g.) have shown that under either certain MPEC hypotheses such as the so-called strong coherent orientation condition (SCOC) or a standard constraint qualification such as the well-known Mangasarian-Fromovitz constraint qualification (MFCQ) for each restricted NLP$_*(\beta_1,\beta_2)$, one can equate $\mathcal{T}(z^*;\mathcal{F})$ with the union of finitely many polyhedral cones. Since in general such a union can not be completely described in a computationally tractable form except by enumeration, this hardship provides the prime motivation for undertaking the present study. Specifically, the results obtained herein improve the previous studies in several major ways: (a) we are able to equate the last two cones in (3) and establish (5); (b) for the second equality in (3), our conditions are both necessary and sufficient; and (c) the latter conditions are constructive in that they provide a finite and practically effective way for checking such equalities. Subsequently, we will also consider a refined representation of $\mathcal{T}(z^*;\mathcal{F})^*$ in order to handle situations where equality fails to hold in (5). As a by-product of the latter investigation, we obtain a necessary and sufficient condition for the cone $\mathcal{T}(z^*;\mathcal{F})$ to be convex.

3 Main Results

The starting point of our analysis is standard NLP theory. Under several well-known constraint qualifications (CQs) in smooth NLP, the tangent cones at z^* of RNLP$_*$ and NLP$_*(\beta_1,\beta_2)$ will coincide with the respective linearization cones of these programs. The most general of these CQs is the Abadie CQ [1] which postulates exactly the latter property for the constraints of these NLPs. We will make this CQ as our first assumption.

(A1) For each pair $(\beta_1,\beta_2)\in\mathcal{P}(\beta)$, the Abadie CQ holds for the NLP$_*(\beta_1,\beta_2)$ at z^*; that is,

$$\begin{aligned}\mathcal{T}(z^*;\mathcal{F}_{(\beta_1,\beta_2)}) \;=\; \{\, dz\in\Re^N \;:\;\; & \nabla g_k(z^*)^T dz \le 0,\; k\in\mathcal{I}\\ & \nabla h_k(z^*)^T dz = 0,\; k=1,\ldots,\ell\\ & \nabla G_j(z^*)^T dz = 0,\; j\in\alpha\cup\beta_1\\ & \nabla G_j(z^*)^T dz \ge 0,\; j\in\beta_2\\ & \nabla H_j(z^*)^T dz \ge 0,\; j\in\beta_1\\ & \nabla H_j(z^*)^T dz = 0,\; j\in\gamma\cup\beta_2\,\};\end{aligned}$$

moreover, the same CQ also holds the RNLP_* at z^*; that is,

$$\begin{aligned}\mathcal{T}(z^*;\mathcal{F}_{\mathrm{R}}) = \{\, dz \in \Re^N : \;& \nabla g_k(z^*)^T dz \le 0,\ k \in \mathcal{I} \\ & \nabla h_k(z^*)^T dz = 0,\ k = 1,\ldots,\ell \\ & \nabla G_j(z^*)^T dz = 0,\ j \in \alpha \\ & \nabla G_j(z^*)^T dz \ge 0,\ j \in \beta \\ & \nabla H_j(z^*)^T dz \ge 0,\ j \in \beta \\ & \nabla H_j(z^*)^T dz = 0,\ j \in \gamma \,\}.\end{aligned}$$

Under the above representation of the tangent cone $\mathcal{T}(z^*;\mathcal{F}_{(\beta_1,\beta_2)})$, we have

$$\begin{aligned}\mathcal{T}(z^*;\mathcal{F}_{(\beta_1,\beta_2)})^* = \{\, v \in \Re^N : v = \;& \sum_{j\in\alpha\cup\beta} \lambda_j \nabla G_j(z^*) + \sum_{j\in\gamma\cup\beta} \mu_j \nabla H_j(z^*) \\ & - \sum_{k\in\mathcal{I}} \eta_k \nabla g_k(z^*) - \sum_{k=1}^{\ell} \delta_k \nabla h_k(z^*), \\ & \lambda_{\beta_2} \ge 0,\ \mu_{\beta_1} \ge 0,\ \eta_{\mathcal{I}} \ge 0 \,\}.\end{aligned} \tag{6}$$

A similar expression can be given for $\mathcal{T}(z^*;\mathcal{F}_{\mathrm{R}})^*$. As Example 1 shows, the cone $\mathcal{T}(z^*;\mathcal{F})$ is not necessarily convex even in the presence of (A1).

Central to questions (I) and (II) is the the notion of a nonsingular inequality in a linear inequality system which we now review [17]. Specifically, given the linear system

$$Ax \le b, \quad Cx = d, \tag{7}$$

an inequality $(Ax \le b)_i$ is said to be *nonsingular* if there exists a feasible solution of the system (7) that satisfies this inequality strictly; an inequality is *singular* if it is not nonsingular. Clearly if $\mathcal{N}$ and $\mathcal{S}$ denote, respectively, the index set of nonsingular and singular inequalities in (7), then this system is equivalent to

$$(\, Ax = b \,)_{\mathcal{S}}, \quad (\, Ax \le b \,)_{\mathcal{N}}, \quad Cx = d.$$

Applying this terminology to the linear inequality representation of $\mathcal{T}(z^*;\mathcal{F}_{\mathrm{R}})$ given in (A1), we let β_{G} denote the subset of β consisting of all indices $i \in \beta$ such that the inequality $\nabla G_i(z^*)^T dz \ge 0$ is nonsingular in the linear inequality system defining $\mathcal{T}(z^*;\mathcal{F}_{\mathrm{R}})$ and let $\beta_{\overline{\mathrm{G}}}$ denote the complement of β_{G} in β. Similarly, we define the complementary index subsets β_{H} and $\beta_{\overline{\mathrm{H}}}$ of β that pertain to $\nabla H_i(z^*)^T dz \ge 0$. In terms of this classification, we note the following

equivalent representation of $\mathcal{T}(z^*;\mathcal{F}_{\rm R})$:

$$\begin{aligned}\mathcal{T}(z^*;\mathcal{F}_{\rm R}) = \{\, dz \in \Re^N : \;& \nabla g_k(z^*)^T dz \le 0,\; k \in \mathcal{I} \\ & \nabla h_k(z^*)^T dz = 0,\; k = 1,\ldots,\ell \\ & \nabla G_j(z^*)^T dz = 0,\; j \in \alpha \cup \beta_{\overline{\rm G}} \\ & \nabla G_j(z^*)^T dz \ge 0,\; j \in \beta_{\rm G} \\ & \nabla H_j(z^*)^T dz \ge 0,\; j \in \beta_{\rm H} \\ & \nabla H_j(z^*)^T dz = 0,\; j \in \gamma \cup \beta_{\overline{\rm H}} \,\}\end{aligned} \tag{8}$$

which yields

$$\begin{aligned}\mathcal{T}(z^*;\mathcal{F}_{\rm R})^* = \{\, v \in \Re^N : v = \;& \sum_{j\in\alpha\cup\beta} \lambda_j \nabla G_j(z^*) + \sum_{j\in\gamma\cup\beta} \mu_j \nabla H_j(z^*) \\ & - \sum_{k\in\mathcal{I}} \eta_k \nabla g_k(z^*) - \sum_{k=1}^{\ell} \delta_k \nabla h_k(z^*), \\ & \lambda_{\beta_{\rm G}} \ge 0,\; \mu_{\beta_{\rm H}} \ge 0,\; \eta_{\mathcal{I}} \ge 0 \,\}.\end{aligned} \tag{9}$$

Furthermore, the representation (8) allows us to establish the following result which identifies a necessary and sufficient condition for equalities to hold throughout (3).

Proposition 1 *Under (A1), equalities hold throughout (3) if and only if $\beta_{\rm G} \cap \beta_{\rm H} = \emptyset$; in this case,*

$$\mathcal{T}(z^*;\mathcal{F}) = \mathcal{T}(z^*;\mathcal{F}_{(\beta_{\overline{\rm G}},\,\beta_{\rm G})}) = \mathcal{T}(z^*;\mathcal{F}_{(\beta_{\overline{\rm H}},\,\beta_{\rm H})}). \tag{10}$$

Proof. Assume $\beta_{\rm G} \cap \beta_{\rm H} = \emptyset$. We claim that

$$\mathcal{T}(z^*;\mathcal{F}_{\rm R}) \subseteq \mathcal{T}(z^*;\mathcal{F}_{(\beta_{\overline{\rm G}},\,\beta_{\rm G})}). \tag{11}$$

Once this is established, it then follows that equalities will hold throughout (3); moreover, the first equality in (10) will also follow. The second equality in (10) follows by a similar argument. To prove (11), note that since $\beta_{\rm G} \cap \beta_{\rm H} = \emptyset$, we have $\beta_{\rm H} \subseteq \beta_{\overline{\rm G}}$; thus comparing the expression (8) with that for $\mathcal{T}(z^*;\mathcal{F}_{(\beta_{\overline{\rm G}},\,\beta_{\rm G})})$, we see that the claimed inclusion must hold.

Conversely, suppose that equalities hold throughout (3). Assume by way of contradiction that there exists an index $i \in \beta_{\rm G} \cap \beta_{\rm H}$. Then there must exist a vector $dz \in \mathcal{T}(z^*;\mathcal{F}_{\rm R})$ such that

$$\nabla G_i(z^*)^T dz > 0 \quad \text{and} \quad \nabla H_i(z^*)^T dz > 0.$$

This vector dz can not belong to $\mathcal{T}(z^*;\mathcal{F}_{(\beta_1,\beta_2)})$ for any $(\beta_1,\beta_2) \in \mathcal{P}(\beta)$, contradicting the assumption. Consequently, we must have $\beta_{\rm G} \cap \beta_{\rm H} = \emptyset$, as desired. Q.E.D.

Since the task of checking singular versus nonsingular inequalities in a linear inequality system can in general be accomplished by solving linear programs, the condition $\beta_{\rm G} \cap \beta_{\rm H} = \emptyset$

can be checked by solving no more than $2|\beta|$ linear programs. Indeed, for each $i \in \beta$, we may check the consistency of the following linear inequality system in the variable dz:

$$\begin{aligned}
\nabla g_k(z^*)^T dz &\leq 0, \quad k \in \mathcal{I} \\
\nabla h_k(z^*)^T dz &= 0, \quad k = 1, \ldots, \ell \\
\nabla G_j(z^*)^T dz &= 0, \quad j \in \alpha \\
\nabla G_j(z^*)^T dz &\geq 0, \quad j \in \beta \setminus \{i\} \\
\nabla G_i(z^*)^T dz &\geq 1 \\
\nabla H_j(z^*)^T dz &\geq 0, \quad j \in \beta \\
\nabla H_j(z^*)^T dz &= 0, \quad j \in \gamma;
\end{aligned}$$

if this system is consistent, then $i \in \beta_{\rm G}$; otherwise $i \in \beta_{\overline{\rm G}}$. In this way, the set $\beta_{\rm G}$ can be completely determined. Next, for each index $i \in \beta_{\rm G}$, we may consider the following linear inequality system in the variable dz:

$$\begin{aligned}
\nabla g_k(z^*)^T dz &\leq 0, \quad k \in \mathcal{I} \\
\nabla h_k(z^*)^T dz &= 0, \quad k = 1, \ldots, \ell \\
\nabla G_j(z^*)^T dz &= 0, \quad j \in \alpha \\
\nabla G_j(z^*)^T dz &\geq 0, \quad j \in \beta \\
\nabla H_j(z^*)^T dz &\geq 0, \quad j \in \beta \setminus \{i\} \\
\nabla H_i(z^*)^T dz &\geq 1 \\
\nabla H_j(z^*)^T dz &= 0, \quad j \in \gamma;
\end{aligned}$$

if this system is consistent, then $i \in \beta_{\rm H}$ and the desired condition fails. Consequently, by means of this constructive procedure, we can effectively check the equalities in (3).

We next examine question (II); that is, when does equality hold in (4)? Before presenting the treatment of this question, we state a fundamental property of the dual cone $\mathcal{T}(z^*; \mathcal{F})^*$ that is perhaps a folklore result but we can not find it formally documented in the literature.

Proposition 2 . *Under (A1), $\mathcal{T}(z^*; \mathcal{F})^*$ is a polyhedral cone.*

Proof. Since each $\mathcal{T}(z^*; \mathcal{F}_{(\beta_1, \beta_2)})$ is a polyhedral cone and $\mathcal{T}(z^*; \mathcal{F})$ is the union of these cones as (β_1, β_2) ranges over the members of $\mathcal{P}(\beta)$, then by noting the following basic facts: (i) the dual of a (nonconvex) set is equal to the dual of the convex hull of the set, (ii) the convex hull of the union of finitely many polyhedral cones is a polyhedral cone, and (iii) the dual of a polyhedral cone is polyhedral, the desired polyhedrality of $\mathcal{T}(z^*; \mathcal{F})^*$ follows easily. Q.E.D.

The implication of the above proposition is that in principle $\mathcal{T}(z^*; \mathcal{F})^*$ can be described in terms of either a system of finitely many homogeneous linear inequalities or as the conical

hull of finitely many vectors; the practical difficulty of course lies in the full description of such inequalities and/or generators. By establishing the equality in (4), we have thus provided one complete description of $\mathcal{T}(z^*;\mathcal{F})^*$.

It turns out that the gradients of the nonsingular "G_i-inequalities" and "H_i-inequalities" for $i \in \beta_{\rm G} \cap \beta_{\rm H}$ have an important role to play in the answer to the question of concern. This role is made precise in the hypothesis (A2) below. For notational convenience, we write

$$\beta_{\rm GH} \equiv \beta_{\rm G} \cap \beta_{\rm H}, \quad \beta_{\overline{\rm GH}} \equiv \beta \setminus \beta_{\rm GH}.$$

Due to Proposition 1, we assume throughout the following discussion that $\beta_{\rm GH}$ is nonempty. We now introduce a preliminary assumption to deal with the second question.

(A2) There exists a pair $(\beta_{\rm GH_1}, \beta_{\rm GH_2}) \in \mathcal{P}(\beta_{\rm GH})$ such that

(A2a) the following gradients at z^*:

$$\mathcal{G}_{\rm dgn} \equiv \{ \nabla G_j(z^*) : j \in \beta_{\rm GH_1} \} \cup \{ \nabla H_j(z^*) : j \in \beta_{\rm GH_2} \}$$

are linearly independent;

(A2b) letting

$$\begin{aligned} \mathcal{G}_{\rm act} \equiv & \{ \nabla G_j(z^*) : j \in \alpha \cup \beta_{\overline{\rm GH}} \cup \beta_{\rm GH_2} \} \cup \{ \nabla H_j(z^*) : j \in \gamma \cup \beta_{\overline{\rm GH}} \cup \beta_{\rm GH_1} \} \\ & \cup \{ \nabla g_k(z^*) : k \in \mathcal{I} \} \cup \{ \nabla h_k(z^*) : k = 1, \ldots, \ell \} \end{aligned}$$

we have

$$\mathrm{span}(\mathcal{G}_{\rm dgn}) \cap \mathrm{span}(\mathcal{G}_{\rm act}) = \{0\}.$$

(By convention, the span of the empty set of vectors is the singleton $\{0\}$.)

We establish an important lemma which gives two equivalent formulations of conditions (A2a) and (A2b). This lemma easily implies that assumption (A2) holds trivially if the linear independence constraint qualification (LICQ) holds for RNLP$_*$ at z^*. The proof of the lemma is based on elementary linear algebraic arguments.

Lemma 1 *Let $(\beta_{\rm GH_1}, \beta_{\rm GH_2}) \in \mathcal{P}(\beta_{\rm GH})$. The three statements below are equivalent:*

(i) (A2a) and (A2b) hold;

(ii) the following implication hold:

$$\sum_{j \in \alpha \cup \beta} \lambda_j \nabla G_j(z^*) + \sum_{j \in \gamma \cup \beta} \mu_j \nabla H_j(z^*) - \sum_{k \in \mathcal{I}} \eta_k \nabla g_k(z^*) - \sum_{k=1}^{\ell} \delta_k \nabla h_k(z^*) = 0 \qquad (12)$$

$$\Rightarrow \begin{cases} \lambda_j = 0, & \forall\, j \in \beta_{\rm GH_1} \\ \mu_j = 0, & \forall\, j \in \beta_{\rm GH_2}. \end{cases}$$

(iii) conditions ($A_{\rm G}$) *and* ($A_{\rm H}$) *below hold:*

($\mathbf{A}_{\rm G}$) *for each* $i \in \beta_{\rm GH_1}$, *there exists a vector* $d_G \in \Re^N$ *such that*

$$\begin{aligned}
&\nabla G_i(z^*)^T d_G \ > \ 0 \\
&\nabla G_j(z^*)^T d_G \ = \ 0, \ \forall\, j \ \in \ \alpha \cup \beta \setminus \{i\} \\
&\nabla H_j(z^*)^T d_G \ = \ 0, \ \forall\, j \ \in \ \gamma \cup \beta \\
&\nabla g_k(z^*)^T d_G \ = \ 0, \ \forall\, k \ \in \ \mathcal{I} \\
&\nabla h_k(z^*)^T d_G \ = \ 0, \ \forall\, k \ = \ 1, \ldots, \ell;
\end{aligned} \tag{13}$$

($\mathbf{A}_{\rm H}$) *for each* $i \in \beta_{\rm GH_2}$, *there exists a vector* $d_H \in \Re^N$ *such that*

$$\begin{aligned}
&\nabla H_i(z^*)^T d_H \ > \ 0 \\
&\nabla H_j(z^*)^T d_H \ = \ 0, \ \forall\, j \ \in \ \gamma \cup \beta \setminus \{i\} \\
&\nabla G_j(z^*)^T d_H \ = \ 0, \ \forall\, j \ \in \ \alpha \cup \beta \\
&\nabla g_k(z^*)^T d_H \ = \ 0, \ \forall\, k \ \in \ \mathcal{I} \\
&\nabla h_k(z^*)^T d_H \ = \ 0, \ \forall\, k \ = \ 1, \ldots, \ell.
\end{aligned} \tag{14}$$

In particular, (A2a) and (A2b) must hold if the gradients in the family

$$\mathcal{G}_{\rm dgn} \cup \mathcal{G}_{\rm act}$$

are linearly independent.

Proof. (i) $\Rightarrow$ (ii) Suppose (12) holds. Then we have

$$-\left(\sum_{j \in \beta_{\rm GH_1}} \lambda_j \nabla G_j(z^*) + \sum_{j \in \beta_{\rm GH_2}} \mu_j \nabla H_j(z^*) \right) =$$

$$\sum_{j \in \alpha \cup \beta_{\overline{\rm GH}} \cup \beta_{\rm GH_2}} \lambda_j \nabla G_j(z^*) + \sum_{j \in \gamma \cup \beta_{\overline{\rm GH}} \cup \beta_{\rm GH_1}} \mu_j \nabla H_j(z^*) - \sum_{k \in \mathcal{I}} \eta_k \nabla g_k(z^*) - \sum_{k=1}^{\ell} \delta_k \nabla h_k(z^*).$$

The vector on the left-hand side belongs to the span of $\mathcal{G}_{\rm dgn}$, whereas the vector on the right-hand side clearly lies in the span of $\mathcal{G}_{\rm act}$. Thus, by (A2b), both vectors must equal to zero. Hence (A2a) implies $\lambda_j = 0, \forall\, j \in \beta_{\rm GH_1}$, and $\mu_j = 0, \forall\, j \in \beta_{\rm GH_2}$. Thus (ii) follows.

(ii) $\Rightarrow$ (iii). Suppose that (ii) holds but that for some index $i \in \beta_{\rm GH_1}$, no vector d_G exists that satisfies (13). It then follows that for some scalars λ_j, μ_j, η_k and δ_k, we have

$$\nabla G_i(z^*) + \sum_{\substack{j \in \alpha \cup \beta \\ j \neq i}} \lambda_j \nabla G_j(z^*) + \sum_{j \in \gamma \cup \beta} \mu_j \nabla H_j(z^*) - \sum_{k \in \mathcal{I}} \eta_k \nabla g_k(z^*) - \sum_{k=1}^{\ell} \delta_k \nabla h_k(z^*) \ = \ 0.$$

Since the left-hand vector has a coefficient of unity associated with it, the above equation clearly contradicts the implication in (ii). Similarly, we can show that a vector d_H satisfying (14) exists for every $i \in \beta_{\mathrm{GH}_2}$. Thus (iii) holds.

(iii) $\Rightarrow$ (i). Suppose that ($\mathrm{A_G}$) and ($\mathrm{A_H}$) hold. If (A2a) fails, then either (a) for some index $i \in \beta_{\mathrm{GH}_1}$, $\nabla G_i(z^*)$ is linearly dependent on the other vectors in the family $\mathcal{G}_{\mathrm{dgn}}$, or (b) for some index $i \in \beta_{\mathrm{GH}_2}$, $\nabla H_i(z^*)$ is linearly dependent on the other vectors in the family $\mathcal{G}_{\mathrm{dgn}}$. Corresponding to such an index i, (a) contradicts ($\mathrm{A_G}$) and (b) contradicts ($\mathrm{A_H}$). Thus (A2a) must hold. If (A2b) fails, then there is a nonzero vector common to the two spans $\mathcal{G}_{\mathrm{dgn}}$ and $\mathcal{G}_{\mathrm{act}}$. Using this common vector and reversing the above argument, one easily contradicts the existence of the vector d_G or d_H corresponding to a certain index $i \in \beta_{\mathrm{GH}}$. Thus (A2b) must hold.

Consequently, the three statements (i), (ii), and (iii) are equivalent. Finally, the last assertion of the lemma is obvious. Q.E.D.

Obviously both (A1) and (A2) are satisfied in Example 1; whereas in Example 2, (A1) is satisfied but (A2) fails to hold. Using Lemma 1, we can establish the following result which provides a sufficient condition for the validity of (5).

Theorem 1 *Under assumptions (A1) and (A2), equality holds in (4); consequently, (5) holds.*

Proof. Let v be a vector in the right-hand set of (4). We claim that $v \in \mathcal{T}(z^*; \mathcal{F}_{\mathrm{R}})^*$. Since

$$\left(\bigcup_{(\beta_1,\beta_2)\in\mathcal{P}(\beta)} \mathcal{T}(z^*; \mathcal{F}_{(\beta_1,\beta_2)}) \right)^* = \bigcap_{(\beta_1,\beta_2)\in\mathcal{P}(\beta)} \mathcal{T}(z^*; \mathcal{F}_{(\beta_1,\beta_2)})^*, \tag{15}$$

we have

$$v \in \mathcal{T}(z^*; \mathcal{F}_{(\beta_1,\beta_2)})^*, \quad \forall\, (\beta_1, \beta_2) \in \mathcal{P}(\beta). \tag{16}$$

Consider the specific pair

$$\beta_1 \equiv \beta_{\mathrm{GH}_1} \cup \beta_{\overline{\mathrm{G}}} \quad \text{and} \quad \beta_2 \equiv \beta_{\mathrm{G}} \setminus \beta_{\mathrm{GH}_1}. \tag{17}$$

Notice that we have

$$\beta_2 = \beta_{\mathrm{GH}_2} \cup (\beta_{\mathrm{G}} \cap \beta_{\overline{\mathrm{H}}}).$$

By the representation of $\mathcal{T}(z^*; \mathcal{F}_{(\beta_1,\beta_2)})^*$ given by (6), it follows that there exist scalars λ_j, μ_j, η_k and δ_k with

$$\begin{aligned} \lambda_j &\geq 0, \quad \forall\, j \in \beta_{\mathrm{G}} \setminus \beta_{\mathrm{GH}_1} \\ \mu_j &\geq 0, \quad \forall\, j \in \beta_{\mathrm{GH}_1} \cup \beta_{\overline{\mathrm{G}}} \\ \eta_k &\geq 0, \quad \forall\, k \in \mathcal{I} \end{aligned} \tag{18}$$

such that

$$v = \sum_{j\in\alpha\cup\beta} \lambda_j \nabla G_j(z^*) + \sum_{j\in\gamma\cup\beta} \mu_j \nabla H_j(z^*) - \sum_{k\in\mathcal{I}} \eta_k \nabla g_k(z^*) - \sum_{k=1}^{\ell} \delta_k \nabla h_k(z^*). \tag{19}$$

In order to show that $v \in \mathcal{T}(z^*; \mathcal{F}_{\mathrm{R}})^*$, it suffices to show, due to the representation (9) of $\mathcal{T}(z^*; \mathcal{F}_{\mathrm{R}})^*$, that

- $\lambda_j \geq 0$ for all $j \in \beta_{\mathrm{GH}_1}$, and
- $\mu_j \geq 0$ for all $j \in \beta_{\mathrm{GH}_2}$,

where the latter is due to the relation

$$\beta_{\mathrm{H}} \;=\; \beta_{\mathrm{GH}_2} \;\cup\; \left[\, \beta_{\mathrm{H}} \cap \left(\beta_{\mathrm{GH}_1} \;\cup\; \beta_{\overline{\mathrm{G}}} \right) \right].$$

We prove only the claim for λ_j because the proof for μ_j is similar. Assume by way of contradiction that $\lambda_i < 0$ for some index $i \in \beta_{\mathrm{GH}_1}$. Corresponding to this index, let d_G be the vector satisfying (13). By (13) and (19), we have

$$v^T d_G \;=\; \lambda_i \nabla G_i(z^*)^T d_G \;<\; 0.$$

Nevertheless, by (A1), this vector d_G belongs to $\mathcal{T}(z^*; \mathcal{F}_{(\beta_1', \beta_2')})$ for any pair $(\beta_1', \beta_2') \in \mathcal{P}(\beta)$ such that $i \in \beta_2'$. Since v belongs to $\mathcal{T}(z^*; \mathcal{F}_{(\beta_1', \beta_2')})^*$ by (16), we must have $v^T d_G \geq 0$ which is a contradiction. Q.E.D.

The proof of Theorem 1 indicates that if (A1) and (A2) are satisfied and if the multipliers $\lambda_j, \mu_j, \eta_k, \delta_k$ satisfying (19) are unique, then for the particular pair (β_1, β_2) in the family $\mathcal{P}(\beta)$ defined by (17) and any pair $(\beta_1', \beta_2') \in \mathcal{P}(\beta)$ such that $\beta_1' \supseteq \beta_{\mathrm{GH}_2}$ and $\beta_2' \supseteq \beta_{\mathrm{GH}_1}$, we have

$$\mathcal{T}(z^*; \mathcal{F})^* \;=\; \mathcal{T}(z^*; \mathcal{F}_{\mathrm{R}})^* \;=\; \mathcal{T}(z^*; \mathcal{F}_{(\beta_1, \beta_2)})^* \;\cap\; \mathcal{T}(z^*; \mathcal{F}_{(\beta_1', \beta_2')})^*. \tag{20}$$

Similar representations can be obtained by reversing the roles of G and H in the above index pairs.

It turns out that the expression (20) will hold under a weakening of assumption (A2). We formally state this weakened assumption below.

(A3) There exists a pair $(\beta_{\mathrm{GH}_1}, \beta_{\mathrm{GH}_2}) \in \mathcal{P}(\beta_{\mathrm{GH}})$ such that

$$\sum_{j \in \alpha \cup \beta} \lambda_j \nabla G_j(z^*) + \sum_{j \in \gamma \cup \beta} \mu_j \nabla H_j(z^*) - \sum_{k \in \mathcal{I}} \eta_k \nabla g_k(z^*) - \sum_{k=1}^{\ell} \delta_k \nabla h_k(z^*) \;=\; 0 \tag{21}$$

$$\Rightarrow \begin{cases} \lambda_j \lambda_{j'} \;\geq\; 0, & \forall\, (j, j') \;\in\; \beta_{\mathrm{GH}_1} \times (\, \beta_{\mathrm{G}} \setminus \beta_{\mathrm{GH}_1} \,) \\ \lambda_j \mu_{j'} \;\geq\; 0, & \forall\, (j, j') \;\in\; \beta_{\mathrm{GH}_1} \times (\, \beta_{\mathrm{H}} \setminus \beta_{\mathrm{GH}_2} \,) \\ \mu_j \mu_{j'} \;\geq\; 0, & \forall\, (j, j') \;\in\; \beta_{\mathrm{GH}_2} \times (\, \beta_{\mathrm{H}} \setminus \beta_{\mathrm{GH}_2} \,) \\ \mu_j \lambda_{j'} \;\geq\; 0, & \forall\, (j, j') \;\in\; \beta_{\mathrm{GH}_2} \times (\, \beta_{\mathrm{G}} \setminus \beta_{\mathrm{GH}_1} \,). \end{cases}$$

Among other things the family $\mathcal{G}_{\mathrm{dgn}}$ of gradient vectors is no longer assumed to be linearly independent. By Lemma 1, (A2) clearly implies (A3).

Theorem 2 *Under assumptions (A1) and (A3), expression (20) holds for the pair* (β_1, β_2) *defined by* $\beta_1 \equiv \beta_{\mathrm{GH}_1} \cup \beta_{\overline{\mathrm{G}}}$ *and* $\beta_2 \equiv \beta_{\mathrm{G}} \setminus \beta_{\mathrm{GH}_1}$, *where* $(\beta_{\mathrm{GH}_1}, \beta_{\mathrm{GH}_2})$ *satisfies (A3), and any pair* $(\beta_1', \beta_2') \in \mathcal{P}(\beta)$ *such that* $\beta_1' \supseteq \beta_{\mathrm{GH}_2}$ *and* $\beta_2' \supseteq \beta_{\mathrm{GH}_1}$.

Proof. In view of (4) and (15), it suffices to show

$$\mathcal{T}(z^*;\mathcal{F}_{(\beta_1,\beta_2)})^* \cap \mathcal{T}(z^*;\mathcal{F}_{(\beta_1',\beta_2')})^* \subseteq \mathcal{T}(z^*;\mathcal{F}_{\rm R})^*.$$

Let v be an arbitrary vector in the left-hand intersection. We claim $v \in \mathcal{T}(z^*;\mathcal{F}_{\rm R})^*$. Since $v \in \mathcal{T}(z^*;\mathcal{F}_{(\beta_1,\beta_2)})^*$, there exist scalars λ_j, μ_j, η_k and δ_k such that (18) and (19) hold. If $\lambda_j \geq 0$ for all $j \in \beta_{\rm GH_1}$ and $\mu_j \geq 0$ for all $j \in \beta_{\rm GH_2}$, then the claim is established by (9). Otherwise, either (a) there exists an index $i \in \beta_{\rm GH_1}$ such that $\lambda_i < 0$ or (b) there exists an index $i \in \beta_{\rm GH_2}$ such that $\mu_i < 0$. Since $v \in \mathcal{T}(z^*;\mathcal{F}_{(\beta_1',\beta_2')})^*$, there exist scalars λ_j', μ_j', η_k' and δ_k' such that

$$\begin{aligned}
\lambda_j' &\geq 0, \quad \forall\, j \in \beta_2' \\
\mu_j' &\geq 0, \quad \forall\, j \in \beta_1' \\
\eta_k' &\geq 0, \quad \forall\, k \in \mathcal{I}
\end{aligned}$$

and

$$v = \sum_{j\in\alpha\cup\beta} \lambda_j' \nabla G_j(z^*) + \sum_{j\in\gamma\cup\beta} \mu_j' \nabla H_j(z^*) - \sum_{k\in\mathcal{I}} \eta_k' \nabla g_k(z^*) - \sum_{k=1}^{\ell} \delta_k' \nabla h_k(z^*). \tag{22}$$

Subtracting (19) from the last equation, we deduce

$$\begin{aligned}
0 = &\sum_{j\in\alpha\cup\beta} (\,\lambda_j' - \lambda_j\,) \nabla G_j(z^*) + \sum_{j\in\gamma\cup\beta} (\,\mu_j' - \mu_j\,) \nabla H_j(z^*) \\
&- \sum_{k\in\mathcal{I}} (\,\eta_k' - \eta_k\,) \nabla g_k(z^*) - \sum_{k=1}^{\ell} (\,\delta_k' - \delta_k\,) \nabla h_k(z^*).
\end{aligned}$$

First consider the case where there exists an index $i \in \beta_{\rm GH_1}$ such that $\lambda_i < 0$. Since $\beta_2' \supseteq \beta_{\rm GH_1}$, we have $\lambda_i' \geq 0$. Consequently, (A3) implies

$$[\,\lambda_j' - \lambda_j \geq 0, \quad \forall\, j \in \beta_{\rm G} \setminus \beta_{\rm GH_1}\,] \quad \text{and} \quad [\,\mu_j' - \mu_j \geq 0, \quad \forall\, j \in \beta_{\rm H} \setminus \beta_{\rm GH_2}\,].$$

By (18), we deduce

$$[\,\lambda_j' \geq \lambda_j \geq 0, \quad \forall\, j \in \beta_{\rm G} \setminus \beta_{\rm GH_1}\,] \quad \text{and} \quad [\,\mu_j' \geq \mu_j \geq 0, \quad \forall\, j \in \beta_{\rm H} \setminus \beta_{\rm GH_2}\,],$$

where the latter is due to

$$\beta_{\rm H} \setminus \beta_{\rm GH_2} \subseteq \beta_{\rm GH_1} \cup \beta_{\overline{\rm G}}.$$

Since

$$[\,\lambda_j' \geq 0, \quad \forall\, j \in \beta_2' \supseteq \beta_{\rm GH_1}\,] \quad \text{and} \quad [\,\mu_j' \geq 0, \quad \forall\, j \in \beta_1' \supseteq \beta_{\rm GH_2}\,]$$

it follows that

$$[\,\lambda_j' \geq 0, \quad \forall\, j \in \beta_{\rm G}\,] \quad \text{and} \quad [\,\mu_j' \geq 0, \quad \forall\, j \in \beta_{\rm H}\,].$$

Therefore (22) and (9) yield $v \in \mathcal{T}(z^*;\mathcal{F}_{\rm R})^*$ as desired. Similarly the same conclusion can be proved in the alternative case where there exists an index $i \in \beta_{\rm GH_2}$ such that $\mu_i < 0$. Q.E.D.

By this time, it should become evident why it is important to classify the singular versus nonsingularity inequalities involving the functions G and H. In the first place, determining such a classification of the inequalities is computationally tractable (albeit not totally trivial) and does not involve complete enumeration; in the second place, the above results clearly suggest the nonsingular G_i-inequalities and H_i-inequalities are the ones that require attention in dealing with $\mathcal{T}(z^*;\mathcal{F})$ and its dual. They are the inequalities that potentially cause this tangent cone to be nonconvex and invalidate any nice representation of its dual. Without a simple description of $\mathcal{T}(z^*;\mathcal{F})^*$, the computational task of verifying B-stationarity in MPEC is in general difficult; this task is made easy with the presence of such properties as (A2a) and (A2b) pertaining to the gradients in the family $\mathcal{G}_{\rm dgn}$. The discussion in the next section further expands the important role of these properties.

Simplified primal-dual characterization of B-stationarity

An important consequence of (10) is a much simplified primal-dual characterization of a B-stationary point of the MPEC (1). Unlike those obtained in [10] whose complexity depends highly on the degenerate set β and involves all partitioning pairs (β_1,β_2) of β (thus are practically very difficult to verify due to the combinatorial curse of enumerating subsets of β), this new characterization says that a feasible solution of an MPEC is B-stationary if and only if it is stationary (in the ordinary sense) for the relaxed NLP at this solution. Scheel and Scholtes [16] attempted to establish this conclusion under the MFCQ on a certain restricted NLP (problem (9) in the reference) at the given solution but made a crucial sign error in their derivation, thus invalidating their result. In contrast, our derivation demonstrates that the simplified characterization is valid under (A1) and (A3), with the latter being the new hypothesis.

Theorem 3 *Let z^* be a feasible solution of MPEC (1) satisfying (A1) and (A3). Then z^* is a B-stationary point of MPEC if and only if there exist vectors $\lambda \in \Re^M$, $\mu \in \Re^M$, $\eta \in \Re^m$ and $\delta \in \Re^\ell$ satisfying the following primal-dual stationarity conditions for $RNLP_*$:*

$$\begin{aligned} 0 &= \nabla_z L(z^*,\lambda,\mu,\eta,\delta) \\ &= \nabla\theta(z^*) - \sum_{j=1}^{M} \left(\lambda_j \nabla G_j(z^*) + \mu_j \nabla H_j(z^*) \right) + \sum_{k=1}^{m} \eta_k \nabla g_k(z^*) + \sum_{k=1}^{\ell} \delta_k \nabla h_k(z^*) \end{aligned}$$

$$\lambda_\alpha \text{ free}, \quad \lambda_\beta \geq 0, \quad \lambda_\gamma = 0$$

$$\mu_\gamma \text{ free}, \quad \mu_\beta \geq 0, \quad \mu_\alpha = 0$$

$$0 \leq \eta \perp g(z^*) \leq 0,$$

where

$$L(z,\lambda,\mu,\eta,\delta) \equiv \theta(z) - \lambda^T G(z) - \mu^T H(z) + \eta^T g(z) + \delta^T h(z)$$

is the MPEC Lagrangian function.

Proof. This requires no proof. Q.E.D.

The significance of the above characterization is twofold. One, the characterization is in the form of a single complementarity system that does not involve the enumeration of subsets of β; two, the result can be applied to MPECs where the first-level constraints involve the second-level variables. Theorem 3 extends some results in [10] in which RNLP$_*$ was also used in the study of optimality conditions for the MPEC; the difference is that the LICQ was assumed therein which is relaxed here.

A complete understanding of the dual of the tangent cone $\mathcal{T}(z^*;\mathcal{F})$ is of primary importance in the study of B-stationarity of an MPEC. Our treatment above has provided an improved understanding of this dual. Specifically, we have identified some sufficient conditions for a simple description of $\mathcal{T}(z^*;\mathcal{F})^*$ in terms of the dual $\mathcal{T}(z^*;\mathcal{F}_{\rm R})^*$ of the relaxed tangent cone $\mathcal{T}(z^*;\mathcal{F}_{\rm R})$. For future research, it would be interesting to obtain a necessary and sufficient condition for these two duals to equal.

C-(non)singular inequalities

The nonsingular G_i-inequalities and H_i-inequalities introduced above are not difficult to identify and have an important role to play in understanding $\mathcal{T}(z^*;\mathcal{F})^*$. Yet these inequalities are not adequate to fully describe the representation of this dual cone in general. We have already observed that Example 2 reveals such an inadequacy; in that example, the presence of the side constraint $g(z) \leq 0$ appears to be the cause of trouble. Another example that demonstrates this inadequacy even in the absence of side constraints is provided by the nonlinear complementarity constraint system in Example 3.1.2 in [10] which we rephrase in terms of the present setting.

Example 3. Consider the case where the functions g and h are absent and

$$\begin{array}{ll} G_1(z) \equiv z_1^2 + z_1 + z_2, & H_1(z) \equiv z_2 \\ G_2(z) \equiv z_1^3 + z_3, & H_2(z) \equiv z_3 \end{array}$$

for $z = (z_1, z_2, z_3) \in \Re^3$. Let $z^* \equiv (0,0,0)$. Thus $\alpha = \gamma = \emptyset$ and $\beta = \{1,2\}$. It is easy to verify that (see [10])

$$\begin{aligned} \mathcal{T}(z^*;\mathcal{F}) &= \{\, (dz_1, dz_2, dz_3) \in \Re^3 : \quad 0 \leq dz_2 \perp dz_1 + dz_2 \geq 0, \\ &\qquad\qquad\qquad\qquad\quad 0 \leq dz_3 \perp dz_3 \geq 0 \,\} \\ &= \{\, (a,0,0) \in \Re^3 : a \geq 0 \,\} \quad \cup \quad \{\, (-a,a,0) \in \Re^3 : a \geq 0 \,\} \end{aligned}$$

which is the union of two half-rays. From this, it follows that

$$\mathcal{T}(z^*;\mathcal{F})^* = \{\, (v_1, v_2, v_3) \in \Re^3 : v_1 \geq 0 \text{ and } v_1 - v_2 \leq 0 \,\}.$$

Yet,

$$\mathcal{T}(z^*;\mathcal{F}_{\rm R}) = \{\, (dz_1, dz_2, dz_3) \in \Re^3 : dz_2 \geq 0, dz_3 \geq 0, \text{ and } dz_1 + dz_2 \geq 0 \,\}$$

and

$$\mathcal{T}(z^*;\mathcal{F}_{\rm R})^* \;=\; \{\,(v_1,v_2,v_3)\;\in\;\Re^3\;:\;v_1\;\geq\;0,\,v_1-v_2\;\leq\;0,\;\;\text{and}\;\;v_3\;\geq\;0\,\};$$

the latter dual is a proper subcone of $\mathcal{T}(z^*;\mathcal{F})^*$. In this example, we have

$$\beta_{\rm G}\;=\;\beta_{\rm H}\;=\;\beta\;=\;\{\,1,2\,\}$$

and there exists no partitioning $(\beta_{{\rm GH}_1},\beta_{{\rm GH}_2})$ of $\beta_{\rm GH}=\{1,2\}$ that satisfies assumption (A3), because (21) holds with $\lambda_1=\mu_1=0$ and arbitrary λ_2 and μ_2 such that $\lambda_2+\mu_2=0$.

The above example and Example 2 both illustrate that the cone $\mathcal{T}(z^*;\mathcal{F}_{\rm R})^*$ can sometimes be too small to completely describe the desired cone $\mathcal{T}(z^*;\mathcal{F})^*$. In order to introduce a larger cone, we consider a shrinkage of $\mathcal{T}(z^*;\mathcal{F}_{\rm R})$. The motivation is derived from the above example where we see that although the constraint $dz_3\geq 0$ is a nonsingular inequality in the inequality system defining $\mathcal{T}(z^*;\mathcal{F}_{\rm R})$, the same inequality is singular in the inequality system defining $\mathcal{T}(z^*;\mathcal{F})$. This observation prompts us to pay special attention to the "C-singular" and "C-nonsingular" inequalities in the cone $\mathcal{T}(z^*;\mathcal{F})$, where "C" signifies "complementarity". Specifically, under (A1), the tangent cone $\mathcal{T}(z^*;\mathcal{F})$ is equal to the set of vectors dz satisfying

$$\left.\begin{array}{ll}
\nabla g_k(z^*)^T dz \;\leq\; 0, & k\;\in\;\mathcal{I}\\
\nabla h_k(z^*)^T dz \;=\; 0, & k\;=\;1,\ldots,\ell\\
\nabla G_j(z^*)^T dz \;=\; 0, & j\;\in\;\alpha\\
\nabla G_j(z^*)^T dz \;\geq\; 0 & \\
\nabla H_j(z^*)^T dz \;\geq\; 0 & \\
(\,\nabla G_j(z^*)^T dz\,)\,(\,\nabla H_j(z^*)^T dz\,) \;=\; 0 & \\
\nabla H_j(z^*)^T dz \;=\; 0, & j\;\in\;\gamma.
\end{array}\right\},\quad j\;\in\;\beta \tag{23}$$

Let $\beta_{\rm CG}$ consist of all indices $i\in\beta$ such that the inequality

$$\nabla G_i(z^*)^T dz\;\geq\;0 \tag{24}$$

is nonsingular in the system (23). Such an inequality is said to be *C-nonsingular*; the inequality (24) is said to be *C-singular* if $i\in\beta\setminus\beta_{\rm CG}$. We define the index set $\beta_{\rm CH}$ of C-nonsingular H_i-inequalities in a similar manner. Notice that we have

$$\beta_{\rm CG}\;\subseteq\;\beta_{\rm G}\quad\text{and}\quad\beta_{\rm CH}\;\subseteq\;\beta_{\rm H}.$$

Also let

$$\beta_{\rm CN}\;\equiv\;\beta_{\rm CG}\;\cap\;\beta_{\rm CH}\quad\text{and}\quad\beta_{\overline{\rm CN}}\;\equiv\;\beta\;\setminus\;\beta_{\rm CN}.$$

In Example 1, we have $\beta_{\rm CN}=\beta_{\rm CG}=\beta_{\rm CH}=\{1\}$ and $\beta_{\overline{\rm CN}}=\emptyset$. In Example 2, we have $\beta_{\rm CG}=\{1\}$, $\beta_{\rm CH}=\emptyset$ and hence $\beta_{\rm CN}=\emptyset$ and $\beta_{\overline{\rm CN}}=\{1\}$. In Example 3, we have $\beta_{\rm CN}=\beta_{\rm CG}=\beta_{\rm CH}=\{1\}$ and $\beta_{\overline{\rm CN}}=\{2\}$.

Having introduced the above refined classification of the inequalities defining $\mathcal{T}(z^*;\mathcal{F})$, we should immediately point out that unlike the indices in $\beta_{\rm G}$ and $\beta_{\rm H}$ that are nonsingular with respect to a standard system of linear inequalities and that can be identified by solving linear programs, the C-nonsingular indices in $\beta_{\rm CG}$ or $\beta_{\rm CH}$ are defined with respect to a (linear) complementarity system. As such, the practical identification of these C-nonsingular inequalities is by no means easy (see however the discussion at the end of this section); indeed, at this time, the only way we know how to identify such an inequality in general is by some form of enumeration (more specifically, by minimizing a linear function subject to linear mixed complementarity constraints). In spite of this computational hardship, we believe that we have identified a sharpened representation of $\mathcal{T}(z^*;\mathcal{F})^*$ (at least conceptually) that can treat cases where equality fails in (4), that is, cases where the RNLP$_*$ is overrelaxing. In addition, it turns out these C-(non)singular inequalities have a central role to play in the convexity of tangent cone $\mathcal{T}(z^*;\mathcal{F})$; see Proposition 3 which shows, among other things, that the void intersection of the C-nonsingular G_i-inequalities and the C-nonsingular H_i-inequalities (i.e., $\beta_{\rm CN} = \emptyset$) provides a necessary and sufficient condition for the cone $\mathcal{T}(z^*;\mathcal{F})$ to be convex. The reader should compare this conclusion with Proposition 1 which pertains to the case where the nonsingular G_i-inequalities do not intersect the nonsingular H_i-inequalities.

In terms of the C-singular and C-nonsingular inequalities, we define the following polyhedral cone:

$$\begin{aligned}
\mathcal{K}_{\rm C} \equiv \{\, dz \in \Re^N : \;& \nabla g_k(z^*)^T dz \le 0,\; k \in \mathcal{I} \\
& \nabla h_k(z^*)^T dz = 0,\; k = 1,\ldots,\ell \\
& \nabla G_j(z^*)^T dz = 0,\; j \in \alpha \cup \beta_{\overline{\rm CG}} \\
& \nabla G_j(z^*)^T dz \ge 0,\; j \in \beta_{\rm CG} \\
& \nabla H_j(z^*)^T dz \ge 0,\; j \in \beta_{\rm CH} \\
& \nabla H_j(z^*)^T dz = 0,\; j \in \gamma \cup \beta_{\overline{\rm CH}} \,\},
\end{aligned} \tag{25}$$

where

$$\beta_{\overline{\rm CG}} \equiv \beta \setminus \beta_{\rm CG} \quad \text{and} \quad \beta_{\overline{\rm CH}} \equiv \beta \setminus \beta_{\rm CH}.$$

Clearly we have

$$\mathcal{T}(z^*;\mathcal{F}) \subseteq \mathcal{K}_{\rm C} \subseteq \mathcal{T}(z^*;\mathcal{F}_{\rm R}). \tag{26}$$

Like the tangent cone of the RNLP$_*$ at z^*, $\mathcal{K}_{\rm C}$ is a relaxation of $\mathcal{T}(z^*;\mathcal{F})$; as such, $\mathcal{K}_{\rm C}$ is not necessarily equal to $\mathcal{T}(z^*;\mathcal{F}_{(\beta_1,\beta_2)})$ for any $(\beta_1,\beta_2) \in \mathcal{P}(\beta)$. Nevertheless, the situation where $\mathcal{K}_{\rm C}$ is equal to $\mathcal{T}(z^*;\mathcal{F}_{(\beta_1,\beta_2)})$ for some $(\beta_1,\beta_2) \in \mathcal{P}(\beta)$ turns out to be particularly significant because it is exactly the case when the original tangent cone $\mathcal{T}(z^*;\mathcal{F})$ is a convex polyhedral cone. This assertion is contained in the following result which confirms the fundamental importance of the C-(non)singular inequalities.

Proposition 3 *Assume that for each pair $(\beta_1,\beta_2) \in \mathcal{P}(\beta)$, $\mathcal{T}(z^*;\mathcal{F}_{(\beta_1,\beta_2)})$ has the polyhedral representation as given by (A1). The following seven statements are equivalent:*

(i) $\beta_{\rm CN} = \emptyset$;

(ii) $\mathcal{T}(z^*;\mathcal{F}) = \mathcal{K}_{\rm C}$;

(iii) $\mathcal{T}(z^*;\mathcal{F})$ *is polyhedral;*

(iv) $\mathcal{T}(z^*;\mathcal{F})$ *is convex;*

(v) for any two elements $dz^i \in \mathcal{T}(z^*,\mathcal{F})$, $i = 1,2$,

$$(\nabla G_j(z^*)^T dz^1)(\nabla H_j(z^*)^T dz^2) = 0, \ \forall\, j \in \beta; \tag{27}$$

(vi) there exists $(\beta_1,\beta_2) \in \mathcal{P}(\beta)$ *such that* $\mathcal{T}(z^*;\mathcal{F}) = \mathcal{T}(z^*;\mathcal{F}_{(\beta_1,\beta_2)})$;

(vii) there exists $(\beta_1,\beta_2) \in \mathcal{P}(\beta)$ *such that* $\mathcal{K}_{\rm C} = \mathcal{T}(z^*;\mathcal{F}_{(\beta_1,\beta_2)})$.

Proof. We establish the following implications:

$$\begin{array}{ccccccccccccc} \text{(i)} & \Rightarrow & \text{(ii)} & \Rightarrow & \text{(iii)} & \Rightarrow & \text{(iv)} & \Leftrightarrow & \text{(v)} & \Rightarrow & \text{(vi)} & \Leftrightarrow & \text{(vii)} \\ & & & & & & \Downarrow & & & & \Downarrow & & \\ & & & & & & \text{(i)} & & & & \text{(iii)}. & & \end{array}$$

(i) $\Rightarrow$ (ii). Suppose that $\beta_{\rm CN} = \beta_{\rm CG} \cap \beta_{\rm CH} = \emptyset$. Let $dz \in \mathcal{K}_{\rm C}$. It then follows from (25) that dz satisfies (23), implying $\mathcal{K}_{\rm C} \subseteq \mathcal{T}(z^*;\mathcal{F})$. Consequently the claim follows from (26).

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). These implications are obvious.

(iv) $\Leftrightarrow$ (v). For any two elements $dz^i \in \mathcal{T}(z^*,\mathcal{F})$, $i = 1,2$ and any scalar $\tau \in [0,1]$, with

$$dz(\tau) \equiv \tau\, dz^1 + (1-\tau)\, dz^2,$$

we have for all indices j,

$$\begin{aligned} &(\nabla G_j(z^*)^T dz(\tau))(\nabla H_j(z^*)^T dz(\tau)) \\ &= \tau(1-\tau)\left[(\nabla G_j(z^*)^T dz^1)(\nabla H_j(z^*)^T dz^2) + (\nabla G_j(z^*)^T dz^2)(\nabla H_j(z^*)^T dz^1)\right]; \end{aligned}$$

moreover, each summand in the right-hand side is nonnegative for all $j \in \beta$. From the identity and the latter remark, the equivalence of (iv) and (v) follows readily.

(iv) $\Rightarrow$ (i). Suppose $\mathcal{T}(z^*;\mathcal{F})$ is convex but there exists an index $i \in \beta_{\rm CN}$. Let dz^G and dz^H be two vectors in $\mathcal{T}(z^*;\mathcal{F})$ such that

$$\nabla G_i(z^*)^T dz^G > 0 \quad \text{and} \quad \nabla H_i(z^*)^T dz^H > 0. \tag{28}$$

By the convexity of $\mathcal{T}(z^*;\mathcal{F})$, for every scalar $\tau \in (0,1)$, the vector

$$dz(\tau) \equiv \tau\, dz^G + (1-\tau)\, dz^H$$

belongs to $\mathcal{T}(z^*;\mathcal{F})$; but this is impossible because

$$\nabla G_i(z^*)^T dz(\tau) > 0 \quad \text{and} \quad \nabla H_i(z^*)^T dz(\tau) > 0,$$

by (28) and the definition of $dz(\tau)$. Consequently, (i) holds.

(v) $\Rightarrow$ (vi). Define the pair (β_1, β_2), where

$$\beta_1 \equiv \beta_{\overline{\mathrm{CG}}} \quad \text{and} \quad \beta_2 \equiv \beta_{\mathrm{CG}}.$$

We claim that $\mathcal{T}(z^*; \mathcal{F}) \subseteq \mathcal{T}(z^*; \mathcal{F}_{(\beta_1,\beta_2)})$. Let $dz \in \mathcal{T}(z^*; \mathcal{F})$ be arbitrary. By the definition of C-singular inequalities, we have $\nabla G_j(z^*)^T dz = 0$ for all $j \in \beta_1 = \beta_{\overline{\mathrm{CG}}}$. On the other hand, by the definition of C-nonsingular inequalities, for each $j \in \beta_2 = \beta_{\mathrm{CG}}$, there exists a vector $dz' \in \mathcal{T}(z^*; \mathcal{F})$ such that $\nabla G_j(z^*)^T dz' > 0$. By (v), it therefore follows that $\nabla H_j(z^*)^T dz = 0$ for all $j \in \beta_2$. This implies that $dz \in \mathcal{T}(z^*; \mathcal{F}_{(\beta_1,\beta_2)})$. Thus the claim is established. Since the reverse inclusion obviously holds, (vi) follows.

(vi) $\Rightarrow$ (iii). This is obvious.

(vi) $\Rightarrow$ (vii). If (vi) holds, then (iii) holds. By the above proofs, it follows that (ii) holds. Thus (vii) also holds.

(vii) $\Rightarrow$ (vi). This is easy because $\mathcal{T}(z^*; \mathcal{F})$ is equal to the union of the tangent cones of the pieces and $\mathcal{K}_{\mathrm{C}}$ is in general a superset of $\mathcal{T}(z^*; \mathcal{F})$. Q.E.D.

Remark. Proposition 3 is reminiscent of the convexity of the solution set of a linear complementarity problem [2, Section 3.5]. In particular, the proof of (v) $\Rightarrow$ (vi) is inspired by a result in [3]; equation (27) expresses the *cross complementarity* property of two arbitrary tangent vectors when the cone $\mathcal{T}(z^*; \mathcal{F})$ is convex.

If β_{CN} is empty, Proposition 3 implies that two cones $\mathcal{T}(z^*; \mathcal{F})$ and $\mathcal{K}_{\mathrm{C}}$, and thus their duals, are equal. We now treat the case where β_{CN} is nonempty by postulating an assumption that is a weakening of (A2). (Remark: the following treatment can similarly been carried out with a weakening of (A3) instead. The details of this refined treatment are omitted.)

(A2′) There exists a pair $(\beta_{\mathrm{CN}_1}, \beta_{\mathrm{CN}_2}) \in \mathcal{P}(\beta_{\mathrm{CN}})$ such that

(A2a′) the following gradients at z^*

$$\mathcal{G}'_{\mathrm{dgn}} \equiv \{ \nabla G_j(z^*) : j \in \beta_{\mathrm{CN}_1} \} \cup \{ \nabla H_j(z^*) : j \in \beta_{\mathrm{CN}_2} \}$$

are linearly independent;

(A2b′) letting

$$\mathcal{G}'_{\mathrm{act}} \equiv \{ \nabla G_j(z^*) : j \in \alpha \cup \beta_{\overline{\mathrm{CN}}} \cup \beta_{\mathrm{CN}_2} \} \cup \{ \nabla H_j(z^*) : j \in \gamma \cup \beta_{\overline{\mathrm{CN}}} \cup \beta_{\mathrm{CN}_1} \}$$

$$\cup \{ \nabla g_k(z^*) : k \in \mathcal{I} \} \cup \{ \nabla h_k(z^*) : k = 1, \ldots, \ell \},$$

we have

$$\mathrm{span}(\mathcal{G}'_{\mathrm{dgn}}) \cap \mathrm{span}(\mathcal{G}'_{\mathrm{act}}) = \{0\}.$$

We note that (A2) $\Rightarrow$ (A2′). Indeed if (A2) holds, then define

$$\beta_{\mathrm{CN}_1} \equiv \beta_{\mathrm{GH}_1} \cap \beta_{\mathrm{CN}} \quad \text{and} \quad \beta_{\mathrm{CN}_2} \equiv \beta_{\mathrm{GH}_2} \cap \beta_{\mathrm{CN}};$$

with this definition of the pair $(\beta_{\mathrm{CN}_1}, \beta_{\mathrm{CN}_2})$, it is not hard to verify (A2a′) and (A2b′), by (A2a) and (A2b). We leave it to the reader to verify that assumption (A2′) is valid for Example 3.

Theorem 4 *Assume that for each pair* $(\beta_1, \beta_2) \in \mathcal{P}(\beta)$, $\mathcal{T}(z^*; \mathcal{F}_{(\beta_1,\beta_2)})$ *has the polyhedral representation as given by (A1). If (A2′) holds, then*

$$\mathcal{T}(z^*; \mathcal{F})^* \;=\; \mathcal{K}_{\rm C}^*.$$

Proof. It suffices to show that

$$\bigcap_{(\beta_1,\beta_2)\in\mathcal{P}(\beta)} \mathcal{T}(z^*; \mathcal{F}_{(\beta_1,\beta_2)})^* \;\subseteq\; \mathcal{K}_{\rm C}^*.$$

As in the proof of Theorem 1, consider an arbitrary vector in the right-hand set and define a specific pair

$$\beta_1 \;\equiv\; \beta_{\rm CN_1} \cup \beta_{\overline{\rm CG}} \quad \text{and} \quad \beta_2 \;\equiv\; \beta_{\rm CG} \setminus \beta_{\rm CN_1}.$$

Note that

$$\beta_2 \;=\; \beta_{\rm CN_2} \cup (\, \beta_{\rm CG} \cap \beta_{\overline{\rm CH}}\,).$$

The remaining part of the proof now resembles that of Theorem 1, by using an equivalent formulation of (A2a′) and (A2b′) that is similar to Lemma 1. The details are not repeated. Q.E.D.

Similarly to Theorem 3, we can state another set of simplified B-stationarity conditions for the MPEC under the hypotheses of Theorem 4.

Theorem 5 *Let* z^* *be a feasible solution of MPEC (1). Assume that for each pair* $(\beta_1, \beta_2) \in \mathcal{P}(\beta)$, $\mathcal{T}(z^*; \mathcal{F}_{(\beta_1,\beta_2)})$ *has the polyhedral representation as given by (A1) and that (A2′) holds. Then* z^* *is a B-stationary point of MPEC if and only if there exist vectors* $\lambda \in \Re^M$, $\mu \in \Re^M$, $\eta \in \Re^m$ *and* $\delta \in \Re^\ell$ *satisfying the following primal-dual conditions:*

$$\begin{aligned} 0 \;&=\; \nabla_z L(z^*, \lambda, \mu, \eta, \delta) \\ &=\; \nabla\theta(z^*) - \sum_{j=1}^{M} (\, \lambda_j \nabla G_j(z^*) + \mu_j \nabla H_j(z^*)\,) + \sum_{k=1}^{m} \eta_k \nabla g_k(z^*) + \sum_{k=1}^{\ell} \delta_k \nabla h_k(z^*) \end{aligned}$$

$$\lambda_{\alpha\cup\beta_{\overline{\rm CG}}} \;\; \textit{free}, \quad \lambda_{\beta_{\rm CG}} \;\geq\; 0, \quad \lambda_\gamma \;=\; 0$$

$$\mu_{\gamma\cup\beta_{\overline{\rm CH}}} \;\; \textit{free}, \quad \mu_{\beta_{\rm CH}} \;\geq\; 0, \quad \mu_\alpha \;=\; 0$$

$$0 \;\leq\; \eta \;\perp\; g(z^*) \;\leq\; 0.$$

We close this section by describing a connection between the C-(non)singular inequalities in an affine complementarity system and the maximal complementarity solutions of such a system. The latter concept was introduced by Güler and Ye [8] who showed that in the context of a standard, monotone linear complementarity problem, these solutions are limit points of interior trajectories. Extending the results in this reference, Ye [21] considered more general affine complementary systems with convex solution sets. The discussion below, which is cast in a context somewhat broader than the tangent structure of an MPEC, is inspired by the work of Ye.

Let Z be an arbitrary subset of $\Re^N$ and G and H be mappings from $\Re^N$ into $\Re^M$. Consider the following constrained complementary system:

$$0 \leq G(z) \perp H(z) \geq 0, \quad z \in Z. \tag{29}$$

For a feasible solution z of this system, we define its *complementarity index* to be the number of strict inequalities in $G(z) \geq 0$ and $H(z) \geq 0$. A solution z is then said to be *maximally complementary* if its complementarity index is maximum. Let $\mathcal{N}_{\rm G}$ be the index set of C-nonsingular G-inequalities in (29); we define $\mathcal{N}_{\rm H}$ similarly.

Proposition 4 *Let G and H be affine maps. Suppose that the solution set of the system (29) is convex. A solution z of the system is maximally complementary if and only if*

$$\{i : G_i(z) > 0\} = \mathcal{N}_{\rm G} \quad \textit{and} \quad \{i : H_i(z) > 0\} = \mathcal{N}_{\rm H}.$$

Proof. Suppose that z is a maximal complementarity solution of (29). Clearly $\{i : G_i(z) > 0\}$ is a subset of $\mathcal{N}_{\rm G}$. If the former is a proper subset of the latter, then there exist a solution z' of (29), distinct from z, and an index i such that $G_i(z') > 0 = G_i(z)$. By considering a convex combination of z and z' and by the linearity property of G and H and the convexity assumption of the solutions of (29), we obtain a solution of (29) that has a larger complementarity index than z; this contradicts the assumed maximality of z. Thus $\{i : G_i(z) > 0\}$ must be equal to $\mathcal{N}_{\rm G}$. Similarly, we can show that $\{i : H_i(z) > 0\}$ is equal to $\mathcal{N}_{\rm H}$.

The converse is fairly obvious and its proof is omitted. Q.E.D.

A practical significance of the above proposition is that for affine G and H (such as in the case of $\mathcal{T}(z^*; \mathcal{F})$), if (29) has a convex solution set, then the C-(non)singular inequalities can be completely determined by finding a single maximal complementarity solution. The reader can consult [8] for the computation of the latter by interior point algorithms.

References

[1] J. Abadie, "On the Kuhn-Tucker theorem", in J. Abadie, ed., *Nonlinear Programming*, North Holland Publishing Company (Amsterdam 1967) pp. 19–36.

[2] R.W. Cottle, J.S. Pang, and R.E. Stone, *The Linear Complementarity Problem*, Academic Press (Boston 1992).

[3] R.W. Cottle, J.S. Pang, and V. Venkateswaran, "Sufficient matrices and the linear complementarity problem", *Linear Algebra and its Applications* 114/115 (1989) 231–249.

[4] F. Facchinei, H. Jiang and L. Qi, "A smoothing method for mathematical programs with equilibrium constraints", *Mathematical Programming*, forthcoming.

[5] J.E. Falk and J. Liu, "On bilevel programming, Part I: General nonlinear case", *Mathematical Programming* 70 (1995) 47-72.

[6] M. Fukushima, Z.Q. Luo and J.S. Pang, "A globally convergent sequential quadratic programming algorithm for mathematical programs with linear complementarity constraints", *Computational Optimization and Applications*, forthcoming.

[7] M. Fukushima and J.S. Pang, "Some feasibility issues in mathematical programs with equilibrium constraints", *SIAM Journal of Optimization*, forthcoming.

[8] O. Güler and Y. Ye, "Convergence behavior of interior-point algorithms", *Mathematical Programming* 60 (1993) 215-228.

[9] H. Jiang and D. Ralph, "QPECgen, a MATLAB generator for mathematical programs with quadratic objectives and affine variational inequality constraints", manuscript, Department of Mathematics, University of Melbourne, Parkville, Australia (December 1997).

[10] Z.Q. Luo, J.S. Pang, and D. Ralph, *Mathematical Programs with Equilibrium Constraints*, Cambridge University Press (Cambridge 1996).

[11] Z.Q. Luo, J.S. Pang, and D. Ralph, "Piecewise sequential quadratic programming for mathematical programs with nonlinear complementarity constraints", in A. Migdalas and P. Pardalos, eds., *Multilevel Optimization: Algorithms, Complexity and Applications*, Kluwer Academic Publishers, forthcoming.

[12] Z.-Q. Luo, J.S. Pang, D. Ralph and S.Q. Wu, "Exact penalization and stationary conditions of mathematical programs with equilibrium constraints", *Mathematical Programming* 76 (1996) 19–76.

[13] J.V. Outrata, "On optimization problems with variational inequality constraints", *SIAM Journal of Optimization* 4 (1994) 340-357.

[14] J.V. Outrata, "Optimization conditions for a class of mathematical programs with equilibrium constraints", manuscript, Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, Prague (revised October 1997).

[15] D. Ralph, "Sequential quadratic programming for mathematical programs with linear complementarity constraints", in R.L. May and A.K. Eastons, eds., *CTAC95 Computational Techniques and Applications*, World Scientific Press (1996) pp. 663-668.

[16] H. Scheel and S. Scholtes, "Mathematical programs with equilibrium constraints: stationarity, optimality, and sensitivity", preprint, University of Cambridge, Department of Engineering & The Judge Institute of Management Studies, Cambridge (September 1997).

[17] J. Stoer and C. Witzgall, *Convexity and Optimization in Finite Dimensions I*, Spring-Verlag (Berlin – Heidelberg 1970)

[18] J.J. Ye, "Optimality conditions for optimization problems with complementarity constraints", manuscript, Department of Mathematics and Statistics, University of Victoria, Victoria (October 1997).

[19] J.J. Ye and X.Y. Ye, "Necessary optimality conditions for optimization problems with variational inequalities constraints", *Mathematics of Operations Research* 22 (1997) 977–998.

[20] J.J. Ye and D.L. Zhu, "Optimality conditions for bilevel programming problems", *Optimization* 33 (1995) 9–27.

[21] Y. Ye, private communication (January 1998).

[22] A. Yezza, "First-order necessary optimality conditions for general bilevel programming problems", *Journal of Optimization Theory and Applications* 89 (1996) 189–219.

[23] R. Zhang, "Problems of hierarchical optimization in finite dimensions", *SIAM Journal on Optimization* 4 (1994) 521-536.