A Globally and Superlinearly Convergent Gauss-Newton Based BFGS Method for Symmetric Equations

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Abstract

In this paper, we present a Gauss-Newton based BFGS method for solving symmetric nonlinear equations which contain, as a special case, an unconstrained optimization problem, a saddle point problem and an equality constrained optimization problem. A suitable line search is proposed with which the presented BFGS method exhibits an approximate norm descent property. Under appropriate conditions, global convergence and superlinear convergence of the method are established.

Key words: BFGS method, global convergence, superlinear convergence, symmetric equations

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1 Introduction

During the past two decades, much effort has been made to establish global convergence of quasi-Newton methods, especially for convex unconstrained minimization problems (e.g. [2], [3], [11], [14], [15], [16], [17]). We refer to [7] for a recent review. For nonconvex minimization problems, the authors [12] recently proposed a modified BFGS method with global and superlinear convergence. For nonlinear equations, to the authors' knowledge, the only global convergence result is due to Griewank [10] for Broyden's rank one method. However, a potential trouble with this method is that the line search may not be executed finitely in a certain special situation [10, pp. 81-82].

The objective of this paper is to present a well defined BFGS method for symmetric equation, which is based on Gauss-Newton method and converges globally and superlinearly under suitable conditions. Unlike traditional quasi-Newton methods for minimization problems, the proposed algorithm is a nonlinear equation based method. So, it may be used to solve such problems as those of finding a stationary point of an unconstrained optimization problem and a KKT point of an equality constrained optimization problem.

The paper is organized as follows. We present a Gauss-Newton based BFGS method for symmetric equations and give some useful properties in the next section. In Section 3, we establish global and superlinear convergence of the proposed method under the assumption of Jacobian uniform nonsingularity. In Section 4, we relax the uniform nonsingularity assumption and extend the results obtained in Section 3.

2 Algorithm and Its Properties

Let $g : R^n \to R^n$ be differentiable. Consider the following system of nonlinear equations:

$$g(x) = 0, \quad x \in R^n. \tag{2.1}$$

We will focus our attention to the case where the Jacobian $\nabla g(x)$ of g is symmetric for all $x \in R^n$. This problem comes, for example, from unconstrained optimization problems and equality constrained problems. When g is the gradient mapping of some function $f : R^n \to R$, (2.1) is just the first order necessary condition for the unconstrained optimization problem

$$\min f(x) \qquad x \in R^n. \tag{2.2}$$

For the equality constrained problem

$$\begin{array}{l} \min f(z) \\ \text{s.t. } h(z) = 0, \end{array} \tag{2.3}$$

where h is a vector-valued function, the KKT conditions can be represented as the system of equations (2.1) with $x = (z, v)$ and

$$g(z,v) = \begin{pmatrix} \nabla f(z) + \nabla h(z)v \\ h(z) \end{pmatrix}, \tag{2.4}$$

where v is the vector of Lagrange multipliers. Notice that the Jacobian $\nabla g(z,v)$ is symmetric for all (z,v). There are some other practical problems such as the saddle point problem, the discretized two-point boundary value problem and discretized elliptic boundary value problem that take the form of (2.1) with symmetric Jacobian.

To begin with, we briefly review Newton's method for solving (2.1). In Newton's method, the subproblem to be solved on each iteration is the following linear equation in p:

$$\nabla g_k^T p + g_k = 0, \tag{2.5}$$

where $g_k = g(x_k)$, $\nabla g_k = \nabla g(x_k)$, and x_k is the current iterate. Define

$$\theta(x) = \frac{1}{2}\|g(x)\|^2. \tag{2.6}$$

Suppose $g_k \neq 0$ and let (2.5) have a solution p_k. If ∇g_k is nonsingular, then we have $p_k^T \nabla \theta(x_k) = -g_k^T \nabla g_k^{-1} \nabla g_k g_k = -\|g_k\|^2 < 0$, which implies that p_k is a descent direction of θ at x_k. In this case, one can choose a steplength $\lambda_k > 0$ satisfying

$$\theta(x_k + \lambda_k p_k) \leq \theta(x_k) + \sigma \lambda_k g_k^T \nabla g_k^T p_k, \tag{2.7}$$

where $\sigma \in (0,1)$ is a given constant. The next iterate is then determined as $x_{k+1} = x_k + \lambda_k p_k$. Newton's method is a norm descent method in the sense that $\|g_{k+1}\| \leq \|g_k\|$ holds on every iteration. It converges globally and quadratically under suitable conditions.

Conventional quasi-Newton methods use a matrix B_k instead of ∇g_k in (2.5). However, if the vector p_k is determined from B_k, then p_k is not necessarily a descent direction of θ at x_k. Therefore the line search (2.7) is no longer usable. One way to globalize such quasi-Newton methods is to employ a new line search rule such as the one given by Griewank [10] for Broyden's method. Another way is to modify a quasi-Newton direction so that it is a descent direction of θ at x_k. In this paper, we adopt the latter strategy.

Notice that, if ∇g_k is nonsingular, (2.5) is equivalent to the system

$$\nabla g_k \nabla g_k^T p + \nabla g_k g_k = 0. \tag{2.8}$$

This means that Newton direction coincides with Gauss-Newton direction

$$p_k = -(\nabla g_k \nabla g_k^T)^{-1} \nabla g_k g_k,$$

which as mentioned above is a descent direction of θ at x_k. We will construct a BFGS method that generates at each iteration an approximate Gauss-Newton direction having an almost

descent property. To do this, notice that if $\nabla g_k^2 = \nabla g_k \nabla g_k^T$ in (2.8) is replaced by matrix B_k, then the solution p_k of (2.8) will be a descent direction of θ at x_k provided that B_k is positive definite. However, even though $\nabla g_k \nabla g_k^T$ is replaced by B_k, (2.8) still contains ∇g_k, the calculation of which should be avoided. To cope with this problem, it will be helpful to observe that if ∇g_k is symmetric, the following relationship holds:

$$g(x_k + \zeta_k g_k) - g_k = \zeta_k \int_0^1 \nabla g(x_k + t\zeta_k g_k) g_k dt,$$

where ζ_k is an arbitrary scalar. Denote $A_k = \int_0^1 \nabla g(x_k + t\zeta_k g_k) dt$. If $\|\zeta_k g_k\|$ is small, then $A_k \approx \nabla g_k$ and hence

$$g(x_k + \zeta_k g_k) - g_k \approx \zeta_k \nabla g_k g_k.$$

Taking this into account, we get the following linear equation as an approximation of (2.8):

$$B_k p + \zeta_k^{-1}[g(x_k + \zeta_k g_k) - g_k] = 0. \tag{2.9}$$

It is not difficult to see that if $\|\zeta_k g_k\|$ is sufficiently small, then the vector p_k obtained by solving (2.9) turns out to be a descent direction of θ at x_k provided that B_k is positive definite. If in addition, B_k is an approximation of $\nabla g_k \nabla g_k^T$, then the solution of (2.9) is an approximate Gauss-Newton direction. However, it is not desirable to use a line search rule like (2.7) because we want to avoid using ∇g_k. Therefore, a new line search technique is needed. In the following, we deal with this problem.

Let $\{\epsilon_k\}$ be a positive sequence satisfying

$$\sum_{k=0}^{\infty} \epsilon_k < \infty. \tag{2.10}$$

We determine a positive steplength λ_k so that the following inequality holds for $\lambda = \lambda_k$:

$$\|g(x_k + \lambda p_k)\|^2 - \|g_k\|^2 \leq -\sigma_1 \|\lambda g_k\|^2 - \sigma_2 \|\lambda p_k\|^2 + \epsilon_k \|g_k\|^2, \tag{2.11}$$

where σ_1 and σ_2 are some positive constants. It is straightforward to see that as $\lambda \to 0^+$, the left-hand side of (2.11) goes to zero, while the right-hand side tends to the positive constant $\epsilon_k \|g_k\|^2$. So, (2.11) is satisfied for all sufficiently small $\lambda > 0$. Hence, we can obtain λ_k by means of a backtracking process.

Now we state a Gauss-Newton based BFGS method for solving (2.1).

Algorithm 1

Step 0. Choose an initial point $x_0 \in R^n$, an initial symmetric positive definite matrix $B_0 \in R^{n \times n}$, a positive sequence $\{\epsilon_k\}$ satisfying (2.10), and constants $r, \rho \in (0,1), \sigma_1, \sigma_2 > 0, \lambda_{-1} > 0$. Let $k := 0$.

Step 1. Solve the following linear equation to get p_k:

$$B_k p + \lambda_{k-1}^{-1}(g(x_k + \lambda_{k-1} g_k) - g_k) = 0. \tag{2.12}$$

Step 2. If

$$\|g(x_k + p_k)\| \leq \rho\|g_k\|, \tag{2.13}$$

then take $\lambda_k = 1$ and go to Step 4. Otherwise go to Step 3.

Step 3. Let i_k be the smallest nonnegative integer i such that (2.11) holds for $\lambda = r^i$. Let $\lambda_k = r^{i_k}$.

Step 4. Let the next iterate be $x_{k+1} = x_k + \lambda_k p_k$.

Step 5. Put $s_k = x_{k+1} - x_k = \lambda_k p_k$, $\delta_k = g_{k+1} - g_k$ and $y_k = g(x_k + \delta_k) - g(x_k)$. If $y_k^T s_k \leq 0$, then $B_{k+1} = B_k$ and go to Step 6. Otherwise, update B_k by BFGS formula:

$$B_{k+1} = B_k - \frac{B_k s_k s_k^T B_k}{s_k^T B_k s_k} + \frac{y_k y_k^T}{y_k^T s_k}. \tag{2.14}$$

Step 6. Let $k := k + 1$. Go to Step 1.

Remark. (1) Recall that B_k approximates $\nabla g_k^2 = \nabla g_k \nabla g_k^T$ rather than ∇g_k. Accordingly, the update formula (2.14) is different from the standard formula in which y_k is the difference of gradients $g_{k+1} - g_k$.

(2) Since Step 5 ensures that B_k is always symmetric and positive definite, (2.12) has a unique solution for each k. Moreover, for every k, Step 3 can be executed in finite steps. Therefore, the method is well defined.

(3) If (2.13) does not hold, then the steplength λ_k satisfies

$$\|g_{k+1}\|^2 - \|g_k\|^2 \leq -\sigma_1\|\lambda_k g_k\|^2 - \sigma_2\|\lambda_k p_k\|^2 + \epsilon_k\|g_k\|^2. \tag{2.15}$$

(4) Since $\{\epsilon_k\}$ satisfies (2.10), the inequalities (2.13) and (2.15) indicate that $\{g_k\}$ is at least approximately norm descent. Moreover, as we will see in Section 3, (2.13) holds for all k sufficiently large. In other words, $\{g_k\}$ is norm descent when k is sufficiently large.

Let Ω be the level set defined by

$$\Omega = \{x \mid \|g(x)\| \leq e^{\frac{\epsilon}{2}}\|g(x_0)\|\}, \tag{2.16}$$

where ϵ is a positive constant such that

$$\sum_{k=0}^{\infty} \epsilon_k \leq \epsilon. \tag{2.17}$$

Then we have the following lemma.

Lemma 2.1 *Let $\{x_k\}$ be generated by Algorithm 1. Then $\{x_k\} \subset \Omega$. Moreover, $\{\|g_k\|\}$ converges.*

Proof For every k, we have either (2.13) or (2.15). In either case, we have $\|g_{k+1}\| \leq (1 + \epsilon_k)^{\frac{1}{2}}\|g_k\| \leq (1 + \epsilon_k)\|g_k\|$. Since ϵ_k satisfies (2.10), we conclude from Lemma 3.4 in [4] that

$\{\|g_k\|\}$ converges. Moreover, we have for all k

$$\begin{aligned}\|g_{k+1}\| &\leq (1+\epsilon_k)^{\frac{1}{2}}\|g_k\| \\ &\vdots \\ &\leq \prod_{i=0}^{k}(1+\epsilon_i)^{\frac{1}{2}}\|g(x_0)\| \\ &\leq e^{\frac{\epsilon}{2}}\|g(x_0)\|,\end{aligned}$$

where ϵ is a constant satisfying (2.17). This implies that $\{x_k\} \subset \Omega$. □

We make the following assumptions which will be frequently used in the rest of this section and the next section.

Assumption A: (i) g is continuously differentiable on an open convex set Ω_1 containing Ω. (ii) The Jacobian of g is symmetric and bounded on Ω_1, i.e. $\nabla g(x)^T = \nabla g(x)$ for every $x \in \Omega_1$ and there exists a positive constant M such that

$$\|\nabla g(x)\| \leq M, \quad \forall x \in \Omega_1.$$

(iii) ∇g is uniformly nonsingular on Ω_1, i.e., there is a constant $m > 0$ such that

$$m\|p\| \leq \|\nabla g(x)p\|, \quad \forall x \in \Omega_1, p \in R^n.$$

Remark The conditions (ii) and (iii) in Assumption A imply that there exist constants $M \geq m > 0$ such that

$$m\|p\| \leq \|\nabla g(x)p\| \leq M\|p\|, \quad \forall x \in \Omega_1, p \in R^n, \tag{2.18}$$

$$\frac{1}{M}\|p\| \leq \|\nabla g(x)^{-1}p\| \leq \frac{1}{m}\|p\|, \quad \forall x \in \Omega_1, p \in R^n \tag{2.19}$$

and

$$m\|x-y\| \leq \|g(x)-g(y)\| \leq M\|x-y\|, \quad \forall x, y \in \Omega_1. \tag{2.20}$$

In particular, for all $x \in \Omega_1$, we have

$$m\|x-x^*\| \leq \|g(x)\| = \|g(x)-g(x^*)\| \leq M\|x-x^*\|, \tag{2.21}$$

where x^* stands for the unique solution of (2.1) in Ω_1. Moreover, the level set Ω is bounded.

Under Assumption A, we can prove some useful properties pertaining to Algorithm 1. The next lemma follows immediately from Assumption A, so that the proof is omitted

Lemma 2.2 *Let the conditions (i) and (ii) in Assumption A be satisfied. Then the following inequalities hold for every k:*

$$\|\delta_k\| \leq M\|s_k\| \quad \textit{and} \quad \|y_k\| \leq M\|\delta_k\| \leq M^2\|s_k\|. \tag{2.22}$$

Denote

$$G_k = \int_0^1 \nabla g(x_k + \tau\lambda_{k-1} g_k) d\tau. \tag{2.23}$$

Then we have $\lambda_{k-1}^{-1}[g(x_k + \lambda_{k-1} g_k) - g_k] = G_k g_k$. Hence (2.12) can be rewritten as

$$B_k p + G_k g_k = 0. \tag{2.24}$$

Lemma 2.3 *Let Assumption A be satisfied. Then the following statements hold:*

(i) If $s_k \to 0$, then there is a constant $m_1 > 0$ such that for all k sufficiently large

$$y_k^T s_k \geq m_1 \|s_k\|^2. \tag{2.25}$$

(ii) Suppose that (2.13) holds only for a finite number of k's. Then we have

$$\sum_{k=0}^{\infty} \|\lambda_k g_k\|^2 < \infty \tag{2.26}$$

and

$$\sum_{k=0}^{\infty} \|\lambda_k p_k\|^2 = \sum_{k=0}^{\infty} \|s_k\|^2 < \infty. \tag{2.27}$$

Moreover, (2.25) holds for all k sufficiently large.

Proof (i) By the mean-value theorem we have

$$\begin{aligned}
y_k^T s_k &= s_k^T [g(x_k + \delta_k) - g_k] \\
&= s_k^T \int_0^1 \nabla g(x_k + \tau\delta_k) d\tau \delta_k \\
&= s_k^T \int_0^1 \nabla g(x_k + \tau\delta_k) d\tau \int_0^1 \nabla g(x_k + \tau s_k) d\tau s_k \\
&= s_k^T \Big[\int_0^1 \nabla g(x_k + \tau s_k) d\tau \Big]^2 s_k \\
&\quad + s_k^T \int_0^1 [\nabla g(x_k + \tau\delta_k) - \nabla g(x_k + \tau s_k)] d\tau \int_0^1 \nabla g(x_k + \tau s_k) d\tau s_k \\
&\geq \Big\| \int_0^1 \nabla g(x_k + \tau s_k) d\tau s_k \Big\|^2 \\
&\quad - \Big\| \int_0^1 [\nabla g(x_k + \tau\delta_k) - \nabla g(x_k + \tau s_k)] d\tau s_k \Big\| \, \Big\| \int_0^1 \nabla g(x_k + \tau s_k) d\tau s_k \Big\| \\
&= \|g_{k+1} - g_k\|^2 - \Big\| \int_0^1 [\nabla g(x_k + \tau\delta_k) - \nabla g(x_k + \tau s_k)] d\tau s_k \Big\| \, \Big\| \int_0^1 \nabla g(x_k + \tau s_k) d\tau s_k \Big\| \\
&\geq m^2 \|s_k\|^2 - M \|s_k\|^2 \int_0^1 \|\nabla g(x_k + \tau\delta_k) - \nabla g(x_k + \tau s_k)\| d\tau. \\
&= \Big[m^2 - M \int_0^1 \|\nabla g(x_k + \tau\delta_k) - \nabla g(x_k + \tau s_k)\| d\tau \Big] \|s_k\|^2,
\end{aligned}$$

where the last inequality follows from (2.18) and (2.20). If $s_k \to 0$, then $\delta_k = g_{k+1} - g_k \to 0$. By the continuity of ∇g, we get (2.25).

(ii) If (2.13) holds for only finitely many k's, then Step 3 is used to determine a steplength λ_k for all k sufficiently large. By (2.15) we have $\sigma_1\|\lambda_k g_k\|^2 + \sigma_2\|s_k\|^2 \leq \|g_k\|^2 - \|g_{k+1}\|^2 + \epsilon_k\|g_k\|^2$. Since $\{\|g_k\|\}$ is bounded and $\{\epsilon_k\}$ satisfies (2.10), we get (2.26) and (2.27) by summing these inequalities. In particular, $\|s_k\| \to 0$, which also implies that (2.25) holds for all k sufficiently large. □

3 Global and Superlinear Convergence

In this section, we establish global and superlinear convergence for Algorithm 1. First, we prove some useful lemmas.

Lemma 3.1 *Suppose that (2.22) and (2.25) hold for all k sufficiently large, say $k \geq k'$. Then there are positive constants β_j, $j = 1,2,3$ such that, for any $k \geq k'$, the inequalities*

$$\beta_2\|s_i\|^2 \leq s_i^T B_i s_i \leq \beta_3\|s_i\|^2 \quad \textit{and} \quad \|B_i s_i\| \leq \beta_1\|s_i\| \tag{3.1}$$

hold for at least a half of indices $i \in \{0,1,2,\ldots,k\}$.

Proof From Theorem 2.1 in [2], conditions (2.25) and (2.22) imply that (3.1) holds for at least $\lceil\frac{k-k'}{2}\rceil$ of indices i such that $k' \leq i \leq k$. Since k' is a fixed integer and B_i are positive definite, we may take smaller β_2, and larger β_1 and β_3 if necessary so that (3.1) holds for all $i < k'$. Therefore (3.1) holds for at least a half of indices $i \in \{0,1,2,\ldots,k\}$. □

The following lemma is important in the analysis of global convergence of Algorithm 1.

Lemma 3.2 *Let the conditions (i) and (ii) in Assumption A hold. If $\lambda_k \neq 1$, then we have the following estimate for λ_k:*

$$\lambda_k \geq \frac{2(p_k^T B_k p_k - t_k\|p_k\|\|g_k\|)r}{\sigma_1\|g_k\|^2 + (\sigma_2 + M^2)\|p_k\|^2}, \tag{3.2}$$

where $t_k = \int_0^1 \|\nabla g(x_k + \tau\lambda_k r^{-1}p_k) - \nabla g(x_k + \tau\lambda_{k-1}g_k)\|d\tau$.

Proof The fact that $\lambda_k \neq 1$ indicates that λ_k was determined in Step 3 and that $\lambda_k' \stackrel{\triangle}{=} \lambda_k/r$ did not satisfy (2.11), i.e.,

$$\begin{aligned}\|g(x_k + \lambda_k' p_k)\|^2 - \|g_k\|^2 &> -\sigma_1\|\lambda_k' g_k\|^2 - \sigma_2\|\lambda_k' p_k\|^2 + \epsilon_k\|g_k\|^2 \\ &\geq -\sigma_1\|\lambda_k' g_k\|^2 - \sigma_2\|\lambda_k' p_k\|^2\end{aligned}$$

or equivalently

$$(\lambda_k')^2(\sigma_1\|g_k\|^2 + \sigma_2\|p_k\|^2) \geq -[\|g(x_k + \lambda_k' p_k)\|^2 - \|g_k\|^2]. \tag{3.3}$$

By an elementary deduction we get

$$\begin{aligned}\|g(x_k+\lambda'_k p_k)\|^2-\|g_k\|^2 &= (g(x_k+\lambda'_k p_k)+g_k)^T(g(x_k+\lambda'_k p_k)-g_k)\\ &= 2g_k^T(g(x_k+\lambda'_k p_k)-g_k)+\|g(x_k+\lambda'_k p_k)-g_k\|^2\\ &\le 2g_k^T(g(x_k+\lambda'_k p_k)-g_k)+M^2\|\lambda'_k p_k\|^2. \qquad (3.4)\end{aligned}$$

For the first term on the right-hand side of (3.4), we have

$$\begin{aligned}&g_k^T(g(x_k+\lambda'_k p_k)-g_k)\\ &= \lambda'_k g_k^T\int_0^1 \nabla g(x_k+\tau\lambda'_k p_k)p_k d\tau\\ &= \lambda'_k g_k^T\int_0^1 \nabla g(x_k+\tau\lambda_{k-1}g_k)p_k d\tau\\ &\quad +\lambda'_k g_k^T\int_0^1 [\nabla g(x_k+\tau\lambda'_k p_k)-\nabla g(x_k+\tau\lambda_{k-1}g_k)d\tau]p_k\\ &= -\lambda'_k p_k^T B_k p_k+\lambda'_k g_k^T\int_0^1 [\nabla g(x_k+\tau\lambda'_k p_k)-\nabla g(x_k+\tau\lambda_{k-1}g_k)]d\tau p_k,\\ &\le -\lambda'_k p_k^T B_k p_k+\lambda'_k\|g_k\|\,\|p_k\|\int_0^1\|\nabla g(x_k+\tau\lambda'_k p_k)-\nabla g(x_k+\tau\lambda_{k-1}g_k)\|d\tau\\ &= -\lambda'_k p_k^T B_k p_k+\lambda'_k t_k\|g_k\|\,\|p_k\|, \qquad (3.5)\end{aligned}$$

where the third equality follows from (2.23) and (2.24). Applying (3.5) to (3.4), we get from (3.3)

$$(\lambda'_k)^2(\sigma_1\|g_k\|^2+\sigma_2\|p_k\|^2)\ge 2\lambda'_k p_k^T B_k p_k-2\lambda'_k t_k\|p_k\|\|g_k\|-M^2\|\lambda'_k p_k\|^2,$$

that is,

$$\lambda'_k\ge\frac{2(p_k^T B_k p_k-t_k\|p_k\|\|g_k\|)}{\sigma_1\|g_k\|^2+(\sigma_2+M^2)\|p_k\|^2}.$$

Since $\lambda_k=\lambda'_k r$, we obtain (3.2). □

Remark It is not difficult to see from the above proof that (3.2) holds true if B_k is updated by any other update formula. In other words, the lemma is independent of the update formula (2.14). We will use this result again in Section 4.

Now we establish a global convergence theorem for Algorithm 1. Let the constants β_j, $j=1,2,3$ be as specified in Lemma 3.1 and define the index set

$$J=\{i\mid (3.1)\text{ holds}\}. \qquad (3.6)$$

Theorem 3.1 *Let Assumption A hold. Then the sequence $\{x_k\}$ generated by Algorithm 1 converges to the unique solution x^* of (2.1).*

Proof By Lemma 2.1, $\{\|g_k\|\}$ converges. So, if

$$\liminf_{k\to\infty}\|g_k\|=0, \qquad (3.7)$$

then every accumulation point of $\{x_k\}$ is a solution of (2.1). Since $\nabla g(x)$ is uniformly non-singular on Ω_1, (2.1) has only one solution. Moreover, since Ω is bounded, $\{x_k\} \subset \Omega$ has at least one accumulation point. Therefore $\{x_k\}$ itself converges to the unique solution of (2.1). It thus suffices to verify (3.7).

If (2.13) holds for infinitely many k's, then (3.7) is trivial. Consider the case where (2.13) holds for only finitely many k's, so that Step 3 is executed for all k sufficiently large. Since (2.26) holds, we only need to show that there is an infinite subsequence of $\{\lambda_k\}$ with a positive lower bound, i.e., $\limsup_{k\to\infty} \lambda_k > 0$.

We assume $\lim_{k\to\infty} \lambda_k = 0$ for contradiction. Notice that $s_k = \lambda_k p_k$ and $B_k s_k = \lambda_k B_k p_k = -\lambda_k G_k g_k$ by (2.24). From (3.1) and (3.6), it is clear that when $k \in J$ is sufficiently large, $p_k^T B_k p_k \geq \beta_2 \|p_k\|^2$ and $\|g_k\| = \|G_k^{-1} B_k p_k\| \leq \frac{1}{m}\|B_k p_k\| \leq \frac{1}{m}\beta_1 \|p_k\|$, where the last inequality follows from (2.19). Since $s_k \to 0$ and $\lambda_k \to 0$, we have $\lambda'_k p_k = r^{-1} s_k \to 0$ and $\lambda_{k-1} g_k \to 0$. Therefore t_k defined in Lemma 3.2 goes to zero. Consequently, we can deduce from (3.2) that $\{\lambda_k\}_{k\in J}$ is bounded away from zero. This contradicts the assumption $\lim_{k\to\infty} \lambda_k = 0$. The proof is complete. □

As a corollary of Theorem 3.1, we have the following result.

Corollary 3.1 *Let Assumption A hold. Then there is a positive constant $\tilde{\lambda}$ such that for every $k \in J$, if (2.13) does not hold, then $\lambda_k \geq \tilde{\lambda}$.*

Proof Since $\{x_k\}$ converges, (2.25) holds for all k sufficiently large and t_k in (3.2) goes to zero. So, it follows from (3.2) that if (2.13) is not true, then λ_k is bounded away from zero for all $k \in J$ large enough, say $k \geq \bar{k}$. Since $\bar{k}$ is a fixed index, we can find a $\tilde{\lambda} > 0$ such that $\lambda_k \geq \tilde{\lambda}$ whenever $k \in J$ and (2.13) does not hold. □

Now we turn to proving the superlinear convergence of Algorithm 1. In the rest of the paper, we abbreviate $g(x^*)$ and $\nabla g(x^*)$ as g_* and ∇g_*, respectively. The following lemma shows that, like ordinary BFGS method, the Dennis-Moré condition ensures superlinear convergence of Algorithm 1. Recall again that B_k is updated so as to approximate $\nabla g_k^2 = \nabla g_k \nabla g_k^T$ in Algorithm 1.

Lemma 3.3 *If*

$$\lim_{k\to\infty} \frac{\|(B_k - \nabla g_*^2) p_k\|}{\|p_k\|} = 0, \tag{3.8}$$

then $\lambda_k \equiv 1$ for all k sufficiently large. Moreover, $\{x_k\}$ converges superlinearly.

Proof Denote

$$\eta_k = \frac{\|(B_k - \nabla g_*^2) p_k\|}{\|p_k\|}. \tag{3.9}$$

Then by (2.24),

$$\|\nabla g_*^2 p_k\| \quad = \quad \|(B_k - \nabla g_*^2) p_k + G_k g_k\|$$

$$
\begin{aligned}
&\leq \|(B_k - \nabla g_*^2)p_k\| + \|G_k g_k\| \\
&\leq \eta_k\|p_k\| + M\|g_k\|, \qquad (3.10)
\end{aligned}
$$

where G_k is defined by (2.23). Since $\eta_k \to 0$, and ∇g_* is nonsingular, the inequality (3.10) implies that there is a constant $M_2 > 0$ such that for all k sufficiently large

$$
\|p_k\| \leq M_2\|g_k\|. \qquad (3.11)
$$

Again by (2.24), we have

$$
\begin{aligned}
\nabla g_*^2(x_k + p_k - x^*) &= \nabla g_*^2(x_k - x^*) + \nabla g_*^2 p_k \\
&= \nabla g_*^2(x_k - x^*) - G_k g_k + (\nabla g_*^2 - B_k)p_k \\
&= \nabla g_*^2(x_k - x^*) - G_k(g_k - g_*) + (\nabla g_*^2 - B_k)p_k \\
&= (\nabla g_*^2 - G_k\overline{G}_k)(x_k - x^*) + (\nabla g_*^2 - B_k)p_k,
\end{aligned}
$$

where $\overline{G}_k = \int_0^1 \nabla g(x^* + \tau(x_k - x^*))d\tau$. It then follows that

$$
\begin{aligned}
\|\nabla g_*^2(x_k + p_k - x^*)\| &\leq \|(\nabla g_*^2 - G_k\overline{G}_k)(x_k - x^*)\| + \|(\nabla g_*^2 - B_k)p_k\| \\
&= \|(\nabla g_*^2 - G_k\overline{G}_k)(x_k - x^*)\| + \eta_k\|p_k\| \\
&\leq \|\nabla g_*^2 - G_k\overline{G}_k\|\|x_k - x^*\| + \eta_k M_2\|g_k\| \\
&\leq \|\nabla g_*^2 - G_k\overline{G}_k\|\|x_k - x^*\| + \eta_k M_2 M\|x_k - x^*\| \\
&= \eta_k M_2 M\|x_k - x^*\| + o(\|x_k - x^*\|), \qquad (3.12)
\end{aligned}
$$

where the second inequality follows from (3.11) and the last inequality follows again from (2.21). Since $\eta_k \to 0$ and ∇g_*^2 is nonsingular, the inequality (3.12) implies

$$
\frac{\|x_k + p_k - x^*\|}{\|x_k - x^*\|} \to 0. \qquad (3.13)
$$

Moreover, we have

$$
\begin{aligned}
\|g(x_k + p_k)\| &= \|g(x_k + p_k) - g_*\| \\
&\leq M\|x_k + p_k - x^*\| \\
&= \frac{M}{m}\frac{\|x_k + p_k - x^*\|}{\|x_k - x^*\|}m\|x_k - x^*\| \\
&\leq \frac{M}{m}\frac{\|x_k + p_k - x^*\|}{\|x_k - x^*\|}\|g_k\|,
\end{aligned}
$$

where the last inequality follows from (2.21). This together with (3.13) indicates that (2.13) is satisfied for all k sufficiently large. In other words, the unit steplength is always accepted for all k sufficiently large. Moreover, (3.13) implies the superlinear convergence of $\{x_k\}$. □

Lemma 3.3 shows that to establish superlinear convergence of Algorithm 1, it suffices to verify that $\{x_k\}$ satisfies the Dennis-Moré condition (3.8).

Lemma 3.4 *Let Assumption A hold. Then, for any fixed $\gamma > 0$, we have*

$$\sum_{k=0}^{\infty} \|x_k - x^*\|^\gamma < \infty. \tag{3.14}$$

Moreover, we have

$$\sum_{k=0}^{\infty} \nu_k(\gamma) < \infty, \tag{3.15}$$

where $\nu_k(\gamma) = \max\{\|x_k - x^*\|^\gamma, \|x_{k+1} - x^*\|^\gamma\}$.

Proof At every iteration i, if the steplength is determined in Step 2, then

$$\|g(x_{i+1})\|^2 \leq \rho^2 \|g(x_i)\|^2. \tag{3.16}$$

Otherwise, we have

$$\|g(x_{i+1})\|^2 \leq (1 - \sigma_1\lambda_i^2 + \epsilon_i)\|g(x_i)\|^2 - \sigma_2\|s_i\|^2 \leq (1 - \sigma_1\lambda_i^2 + \epsilon_i)\|g(x_i)\|^2. \tag{3.17}$$

Moreover, from Corollary 3.1 we see that for each $i \in J$, either (3.16) holds or $\lambda_i \geq \tilde{\lambda}$ for some constant $\tilde{\lambda} > 0$. In the latter case, it follows from (3.17) that

$$\|g(x_{i+1})\|^2 \leq (1 + \epsilon_i - \sigma_1\tilde{\lambda}^2)\|g(x_i)\|^2.$$

Since $\epsilon_i \to 0$, there are an index i' and a constant $\rho_1 \in (0,1)$ such that $1 + \epsilon_i - \sigma_1\tilde{\lambda}^2 \leq \rho_1$ holds for all $i \geq i'$. Let $\rho_2 = \min\{\rho^2, \rho_1\} < 1$. Then

$$\|g(x_{i+1})\|^2 \leq \rho_2 \|g(x_i)\|^2 \tag{3.18}$$

holds for for all $i \in J$ with $i \geq i'$. Lemma 3.1 says that the number of elements in J is at least $\lceil \frac{k}{2} \rceil$. Therefore, for any $k > 2i'$, there are at least $\lceil \frac{k}{2} \rceil - i'$ of indices i such that (3.18) holds. Let K denote the set of indices i for which (3.18) holds. Also, let r_k denote the number of indices in K not exceeding k. Then we have $r_k \geq \lceil \frac{k}{2} \rceil - i'$ for each k. Multiplying (3.17) for $i \notin K$ and (3.18) for $i \in K$ from $i = i'$ to $i = k$ yields

$$\begin{aligned}
\|g_{k+1}\|^2 &\leq \Big[\prod_{\substack{i \\ i \notin K}}^{k} (1 + \epsilon_i - \sigma_1\lambda_i^2) \Big] \rho_2^{r_k} \|g(x_{i'})\|^2 \\
&\leq \Big[\prod_{\substack{i \\ i \notin K}}^{k} (1 + \epsilon_i) \Big] \rho_2^{r_k} \|g(x_{i'})\|^2 \\
&\leq \Big[\prod_{i=0}^{k} (1 + \epsilon_i) \Big] \rho_2^{\lceil \frac{k}{2} \rceil - i'} \|g(x_{i'})\|^2 \\
&\leq e^{\epsilon} \rho_2^{\frac{k}{2} - (i'+1)} \|g(x_{i'})\|^2 \\
&= c' \tilde{\rho}^k,
\end{aligned}$$

where $c' = e^{\epsilon}\rho_2^{-(i'+1)}\|g(x_{i'})\|^2$ and $\tilde{\rho} = \rho_2^{\frac{1}{2}} \in (0,1)$. This together with (2.21) shows that $\|x_{k+1} - x^*\|^2 \leq m^{-2}c'\tilde{\rho}^k$ holds for all k large enough. Hence we have (3.14) for any γ.

Since $\nu_k(\gamma) \leq \|x_{k+1} - x^*\|^\gamma + \|x_k - x^*\|^\gamma$, (3.15) follows from (3.14). □

To obtain superlinear convergence of Algorithm 1, we need a further assumption on g.

Assumption B ∇g is Hölder continuous at x^*, i.e., there are positive constants M_3 and α such that for every x in a neighbourhood of x^*

$$\|\nabla g(x) - \nabla g(x^*)\| \leq M_3\|x - x^*\|^\alpha. \tag{3.19}$$

Lemma 3.5 *Let Assumptions A and B hold. Then there exist positive constants M_4 and m_2 such that for all k sufficiently large*

$$\|y_k - \nabla g_*^2 s_k\| \leq M_4\nu_k\|s_k\| \quad \textit{and} \quad \|y_k\| \geq m_2\|s_k\|, \tag{3.20}$$

where $\nu_k = \max\left\{\|x_k - x^*\|^\alpha, \|x_{k+1} - x^*\|^\alpha\right\}$.

Proof Since $x_k \to x^*$, (3.19) holds for all k large enough. By the mean value theorem we have for all k sufficiently large

$$\begin{aligned}
\|y_k - \nabla g_*^2 s_k\| &= \|g(x_k + \delta_k) - g_k - \nabla g_*^2 s_k\| \\
&= \Big\| \int_0^1 \nabla g(x_k + \tau\delta_k)d\tau\delta_k - \nabla g_*^2 s_k \Big\| \\
&= \Big\| \Big[\int_0^1 \nabla g(x_k + \tau\delta_k)d\tau \int_0^1 \nabla g(x_k + ts_k)dt - \nabla g_*^2 \Big] s_k \Big\| \\
&\leq \Big\| \int_0^1 (\nabla g(x_k + \tau\delta_k) - \nabla g_*)d\tau \int_0^1 \nabla g(x_k + ts_k)dt \Big\| \|s_k\| \\
&\quad + \Big\| \nabla g_* \int_0^1 (\nabla g(x_k + ts_k) - \nabla g_*)dt \Big\| \|s_k\| \\
&\leq M\|s_k\| \Big[\int_0^1 \|\nabla g(x_k + \tau\delta_k) - \nabla g_*\|d\tau \\
&\quad + \int_0^1 \|\nabla g(x_k + ts_k) - \nabla g_*\|dt \Big] \\
&\leq MM_3\|s_k\| \Big[\int_0^1 \|x_k - x^* + \tau\delta_k\|^\alpha d\tau + \int_0^1 \|x_k - x^* + ts_k\|^\alpha dt \Big] \\
&\leq MM_3\|s_k\| \Big[\int_0^1 (\|x_k - x^*\| + \|\delta_k\|)^\alpha d\tau + \int_0^1 \|(1-t)x_k + tx_{k+1} - x^*\|^\alpha dt \Big] \\
&\leq MM_3\|s_k\| \Big\{ (\|x_k - x^*\| + \|\delta_k\|)^\alpha \\
&\quad + \int_0^1 ((1-t)\|x_k - x^*\| + t\|x_{k+1} - x^*\|)^\alpha dt \Big\} \\
&\leq MM_3\|s_k\| \{(\|x_k - x^*\| + M\|s_k\|)^\alpha + \nu_k\} \\
&\leq MM_3\|s_k\| \{[\|x_k - x^*\| + M(\|x_k - x^*\| + \|x_{k+1} - x^*\|)]^\alpha + \nu_k\} \\
&\leq M_4\nu_k\|s_k\|,
\end{aligned} \tag{3.21}$$

where $M_4 = MM_3[(2M+1)^\alpha + 1]$. This yields the first inequality of (3.20). Moreover, by (3.21) we have

$$\|y_k\| \geq \|\nabla g_*^2 s_k\| - \|y_k - \nabla g_*^2 s_k\| \geq \|\nabla g_*^2 s_k\| - M_4 \nu_k \|s_k\|.$$

Since ∇g_* is nonsingular and $\nu_k \to 0$, the second inequality of (3.20) follows from the last inequality. □

Denote $P = \nabla g_*^{-1}$. For an $n \times n$ matrix A, define a matrix norm $\|A\|_P = \|PAP\|_F$, where $\|\cdot\|_F$ denotes the Frobenius norm of a matrix. We let H_k and H_{k+1} stand for the inverse matrices of B_k and B_{k+1}, respectively.

We notice that Theorem 3.1 ensures that $\{x_k\}$ converges. In particular, $s_k \to 0$. Therefore Lemma 2.3 (i) yields that $y_k^T s_k > 0$ for all k sufficiently large. Hence we see from Step 5 in Algorithm 1 that for all k large enough, B_{k+1} is always generated by the update formula (2.14).

The following lemma shows that the BFGS formula (2.14) exhibits a similar property to that of the ordinary BFGS formula.

Lemma 3.6 *Under Assumptions A and B, there exist positive constants M_5, M_6, M_7 and $\beta \in (0,1)$ such that for all k sufficiently large*

$$\|B_{k+1} - \nabla g_*^2\|_P \leq \|B_k - \nabla g_*^2\|_P + M_5 \nu_k, \tag{3.22}$$

$$\|H_{k+1} - \nabla g_*^{-2}\|_{P^{-1}} \leq (1 - \tfrac{1}{2}\beta\mu_k^2 + M_6\nu_k)\|H_k - \nabla g_*^{-2}\|_{P^{-1}} + M_7\nu_k, \tag{3.23}$$

where $\nu_k = \max\left\{\|x_k - x^*\|^\alpha, \|x_{k+1} - x^*\|^\alpha\right\}$ and μ_k is given by

$$\mu_k = \frac{\|P^{-1}[H_k - \nabla g_*^{-2}]y_k\|}{\|H_k - \nabla g_*^{-2}\|_{P^{-1}}\|Py_k\|}. \tag{3.24}$$

In particular, $\{\|B_k\|\}$ and $\{\|H_k\|\}$ are bounded.

Proof From the update formula (2.14), we get

$$\begin{aligned} P(B_{k+1} - \nabla g_*^2)P &= P(B_k - \nabla g_*^2)P \\ &\quad - \frac{[(PB_kP)(P^{-1}s_k)][(PB_kP)(P^{-1}s_k)]^T}{(P^{-1}s_k)^T(PB_kP)(P^{-1}s_k)} \\ &\quad + \frac{(Py_k)(Py_k)^T}{(Py_k)^T(P^{-1}s_k)}. \end{aligned} \tag{3.25}$$

Denote $\tilde{B}_k = PB_kP$, $\tilde{s}_k = P^{-1}s_k$, $\tilde{y}_k = Py_k$, $Q_{k+1} = P(B_{k+1} - \nabla g_*^2)P = PB_{k+1}P - I$, and $Q_k = P(B_k - \nabla g_*^2)P = \tilde{B}_k - I$. Then Q_k, Q_{k+1} and $\tilde{B}_k$ are symmetric, and (3.25) is rewritten as

$$Q_{k+1} = Q_k - \frac{\tilde{B}_k\tilde{s}_k\tilde{s}_k^T\tilde{B}_k}{\tilde{s}_k^T\tilde{B}_k\tilde{s}_k} + \frac{\tilde{y}_k\tilde{y}_k^T}{\tilde{y}_k^T\tilde{s}_k}.$$

Taking the norm operation on both sides, we get

$$\|Q_{k+1}\|_F \le \Big\|Q_k - \frac{\tilde{B}_k\tilde{s}_k\tilde{s}_k^T\tilde{B}_k}{\tilde{s}_k^T\tilde{B}_k\tilde{s}_k} + \frac{\tilde{s}_k\tilde{s}_k^T}{\|\tilde{s}_k\|^2}\Big\|_F + \Big\|\frac{\tilde{s}_k\tilde{s}_k^T}{\|\tilde{s}_k\|^2} - \frac{\tilde{y}_k\tilde{y}_k^T}{\tilde{y}_k^T\tilde{s}_k}\Big\|_F. \tag{3.26}$$

We estimate the two terms on the right-hand side of (3.26). For the first term, we have

$$\begin{aligned}
&\Big\|Q_k - \frac{\tilde{B}_k\tilde{s}_k\tilde{s}_k^T\tilde{B}_k}{\tilde{s}_k^T\tilde{B}_k\tilde{s}_k} + \frac{\tilde{s}_k\tilde{s}_k^T}{\|\tilde{s}_k\|^2}\Big\|_F^2 \\
&\quad= \mathrm{trace}\Big\{\Big(Q_k - \frac{\tilde{B}_k\tilde{s}_k\tilde{s}_k^T\tilde{B}_k}{\tilde{s}_k^T\tilde{B}_k\tilde{s}_k} + \frac{\tilde{s}_k\tilde{s}_k^T}{\|\tilde{s}_k\|^2}\Big)\Big(Q_k - \frac{\tilde{B}_k\tilde{s}_k\tilde{s}_k^T\tilde{B}_k}{\tilde{s}_k^T\tilde{B}_k\tilde{s}_k} + \frac{\tilde{s}_k\tilde{s}_k^T}{\|\tilde{s}_k\|^2}\Big)^T\Big\} \\
&\quad= \mathrm{trace}\Big(Q_k^2 + \frac{\|\tilde{B}_k\tilde{s}_k\|^2\tilde{B}_k\tilde{s}_k\tilde{s}_k^T\tilde{B}_k}{(\tilde{s}_k^T\tilde{B}_k\tilde{s}_k)^2} + \frac{\tilde{s}_k\tilde{s}_k^T}{\|\tilde{s}_k\|^2} - \frac{Q_k\tilde{B}_k\tilde{s}_k\tilde{s}_k^T\tilde{B}_k + \tilde{B}_k\tilde{s}_k\tilde{s}_k^T\tilde{B}_kQ_k}{\tilde{s}_k^T\tilde{B}_k\tilde{s}_k} \\
&\qquad - \frac{\tilde{B}_k\tilde{s}_k\tilde{s}_k^T + \tilde{s}_k\tilde{s}_k^T\tilde{B}_k}{\|\tilde{s}_k\|^2} + \frac{Q_k\tilde{s}_k\tilde{s}_k^T + \tilde{s}_k\tilde{s}_k^TQ_k}{\|\tilde{s}_k\|^2}\Big) \\
&\quad= \|Q_k\|_F^2 + \frac{\|\tilde{B}_k\tilde{s}_k\|^4}{(\tilde{s}_k^T\tilde{B}_k\tilde{s}_k)^2} + 1 - 2\frac{\tilde{s}_k^T\tilde{B}_kQ_k\tilde{B}_k\tilde{s}_k}{\tilde{s}_k^T\tilde{B}_k\tilde{s}_k} - 2\frac{\tilde{s}_k^T\tilde{B}_k\tilde{s}_k}{\|\tilde{s}_k\|^2} + 2\frac{\tilde{s}_k^TQ_k\tilde{s}_k}{\|\tilde{s}_k\|^2}.
\end{aligned} \tag{3.27}$$

By the definition of Q_k and $\tilde{B}_k$, we have

$$\begin{aligned}
\tilde{s}_k^T\tilde{B}_kQ_k\tilde{B}_k\tilde{s}_k &= \tilde{s}_k^T\tilde{B}_k(\tilde{B}_k - I)\tilde{B}_k\tilde{s}_k \\
&= \tilde{s}_k^T\tilde{B}_k^3\tilde{s}_k - \tilde{s}_k^T\tilde{B}_k^2\tilde{s}_k \\
&= \tilde{s}_k^T\tilde{B}_k^3\tilde{s}_k - \|\tilde{B}_k\tilde{s}_k\|^2
\end{aligned}$$

and

$$\tilde{s}_k^TQ_k\tilde{s}_k = \tilde{s}_k^T\tilde{B}_k\tilde{s}_k - \|\tilde{s}_k\|^2.$$

So, we get from (3.27)

$$\begin{aligned}
&\Big\|Q_k - \frac{\tilde{B}_k\tilde{s}_k\tilde{s}_k^T\tilde{B}_k}{\tilde{s}_k^T\tilde{B}_k\tilde{s}_k} + \frac{\tilde{s}_k\tilde{s}_k^T}{\|\tilde{s}_k\|^2}\Big\|_F^2 \\
&= \|Q_k\|_F^2 + \frac{\|\tilde{B}_k\tilde{s}_k\|^4}{(\tilde{s}_k^T\tilde{B}_k\tilde{s}_k)^2} + 1 - 2\frac{\tilde{s}_k^T\tilde{B}_k^3\tilde{s}_k}{\tilde{s}_k^T\tilde{B}_k\tilde{s}_k} + 2\frac{\|\tilde{B}_k\tilde{s}_k\|^2}{\tilde{s}_k^T\tilde{B}_k\tilde{s}_k} - 2\frac{\tilde{s}_k^T\tilde{B}_k\tilde{s}_k}{\|\tilde{s}_k\|^2} + 2\frac{\tilde{s}_k^T\tilde{B}_k\tilde{s}_k}{\|\tilde{s}_k\|^2} - 2 \\
&= \|Q_k\|_F^2 + 2\Big[\Big(\frac{\|\tilde{B}_k\tilde{s}_k\|^2}{\tilde{s}_k^T\tilde{B}_k\tilde{s}_k}\Big)^2 - \frac{\tilde{s}_k^T\tilde{B}_k^3\tilde{s}_k}{\tilde{s}_k^T\tilde{B}_k\tilde{s}_k}\Big] - \Big[\Big(\frac{\|\tilde{B}_k\tilde{s}_k\|^2}{\tilde{s}_k^T\tilde{B}_k\tilde{s}_k}\Big)^2 - 2\frac{\|\tilde{B}_k\tilde{s}_k\|^2}{\tilde{s}_k^T\tilde{B}_k\tilde{s}_k} + 1\Big] \\
&= \|Q_k\|_F^2 + 2\Big[\Big(\frac{\|\tilde{B}_k\tilde{s}_k\|^2}{\tilde{s}_k^T\tilde{B}_k\tilde{s}_k}\Big)^2 - \frac{\tilde{s}_k^T\tilde{B}_k^3\tilde{s}_k}{\tilde{s}_k^T\tilde{B}_k\tilde{s}_k}\Big] - \Big[1 + \frac{\|\tilde{B}_k\tilde{s}_k\|^2}{\tilde{s}_k^T\tilde{B}_k\tilde{s}_k}\Big]^2 \\
&\le \|Q_k\|_F^2,
\end{aligned} \tag{3.28}$$

where the inequality holds because

$$\|\tilde{B}_k\tilde{s}_k\|^2 = [\tilde{B}_k^{\frac{3}{2}}\tilde{s}_k]^T[\tilde{B}_k^{\frac{1}{2}}\tilde{s}_k] \le \|\tilde{B}_k^{\frac{3}{2}}\tilde{s}_k\|\|\tilde{B}_k^{\frac{1}{2}}\tilde{s}_k\| = [\tilde{s}_k^T\tilde{B}_k^3\tilde{s}_k]^{\frac{1}{2}}[\tilde{s}_k^T\tilde{B}_k\tilde{s}_k]^{\frac{1}{2}}.$$

For the second term of (3.26) we have

$$\begin{aligned}
\Big\|\frac{\tilde{s}_k\tilde{s}_k^T}{\|\tilde{s}_k\|^2}-\frac{\tilde{y}_k\tilde{y}_k^T}{\tilde{y}_k^T\tilde{s}_k}\Big\|_F &\le \Big\|\Big(\frac{1}{\|\tilde{s}_k\|^2}-\frac{1}{\tilde{y}_k^T\tilde{s}_k}\Big)\tilde{s}_k\tilde{s}_k^T\Big\|_F+\Big\|\frac{\tilde{s}_k\tilde{s}_k^T-\tilde{y}_k\tilde{y}_k^T}{\tilde{y}_k^T\tilde{s}_k}\Big\|_F\\
&\le \frac{|\tilde{y}_k^T\tilde{s}_k-\|\tilde{s}_k\|^2|}{\tilde{y}_k^T\tilde{s}_k}+\frac{\|\tilde{s}_k(\tilde{s}_k-\tilde{y}_k)^T\|_F+\|(\tilde{s}_k-\tilde{y}_k)\tilde{y}_k^T\|_F}{\tilde{y}_k^T\tilde{s}_k}\\
&\le \frac{\|\tilde{y}_k-\tilde{s}_k\|\,\|\tilde{s}_k\|+\|\tilde{s}_k\|\,\|\tilde{s}_k-\tilde{y}_k\|+\|\tilde{s}_k-\tilde{y}_k\|\,\|\tilde{y}_k\|}{\tilde{y}_k^T\tilde{s}_k}\\
&= \frac{(2\|P^{-1}s_k\|+\|Py_k\|)\|Py_k-P^{-1}s_k\|}{y_k^Ts_k}\\
&\le \frac{(2\|P^{-1}\|\,\|s_k\|+\|P\|\,\|y_k\|)\|P\|\,\|y_k-\nabla g_*^2s_k\|}{y_k^Ts_k}\\
&\le \frac{1}{m_1}(2\|P^{-1}\|+\|P\|M^2)\|P\|M_4\nu_k\\
&= M_5\nu_k, \qquad (3.29)
\end{aligned}$$

where $M_5=\frac{1}{m_1}(2\|P^{-1}\|+\|P\|M^2)\|P\|M_4$ and the last inequality follows from (2.22), (2.25) and (3.20). Since $\|Q\|_F=\|B_k-\nabla g_k^2\|_P$ and $\|Q_{k+1}\|_F=\|B_{k+1}-\nabla g_k^2\|_P$, (3.22) follows from (3.26), (3.28) and (3.29).

Now we verify (3.23). The inverse update formula of BFGS method is represented as

$$\begin{aligned}
H_{k+1} &= H_k+\frac{(s_k-H_ky_k)s_k^T+s_k(s_k-H_ky_k)^T}{y_k^Ts_k}-\frac{y_k^T(s_k-H_ky_k)s_ks_k^T}{(y_k^Ts_k)^2}\\
&= \Big(I-\frac{s_ky_k^T}{y_k^Ts_k}\Big)H_k\Big(I-\frac{y_ks_k^T}{y_k^Ts_k}\Big)+\frac{s_ks_k^T}{y_k^Ts_k},
\end{aligned}$$

which is the dual form of DFP update formula in the sense that $H_k\leftrightarrow B_k$, $H_{k+1}\leftrightarrow B_{k+1}$ and $s_k\leftrightarrow y_k$. Also, from (2.19) and (3.20) we deduce that

$$\|Py_k\|\ge\frac{1}{M}\|y_k\|\ge\frac{m_2}{M}\|s_k\|, \qquad (3.30)$$

$$\|Py_k-P^{-1}s_k\|\le\|P\|\|y_k-\nabla g_*^2s_k\|\le M_4\nu_k\|P\|\,\|s_k\|, \qquad (3.31)$$

and

$$\|s_k-\nabla g_*^{-2}y_k\|\le\|P^2\|\,\|y_k-\nabla g_*^2s_k\|\le M_4\nu_k\|P^2\|\,\|s_k\|. \qquad (3.32)$$

Since $\nu_k\to 0$, the condition of Lemma 3.1 in [4] is satisfied (with the identification $s\leftrightarrow y_k$, $y\leftrightarrow s_k$, $B\leftrightarrow H_k$, $A\leftrightarrow\nabla g_*^{-2}$ and $M\leftrightarrow P^{-1}$). Therefore there are constants $\alpha_1>0,\alpha_2>0$ and $\beta\in(0,1)$ such that

$$\begin{aligned}
\|H_{k+1}-\nabla g_*^{-2}\|_{P^{-1}} &\le \Big(\sqrt{1-\beta\mu_k^2}+\alpha_1\frac{\|P^{-1}s_k-Py_k\|}{\|Py_k\|}\Big)\|H_k-\nabla g_*^{-2}\|_{P^{-1}}\\
&\quad+\alpha_2\frac{\|s_k-\nabla g_*^{-2}y_k\|}{\|Py_k\|}, \qquad (3.33)
\end{aligned}$$

where μ_k is defined by (3.24). Notice that $\sqrt{1-\beta\mu_k^2} \le 1 - \frac{1}{2}\beta\mu_k^2$. In view of (3.30), (3.31) and (3.32), we get (3.23).

Finally, by Lemma 3.4 in [4] and (3.14), we see that $\{\|B_k - \nabla g_*^2\|_P\}$ and $\{\|H_k - \nabla g_*^{-2}\|_{P^{-1}}\}$ converge. In particular, $\{\|B_k\|\}$ and $\{\|H_k\|\}$ are bounded. □

Now we prove the superlinear convergence theorem for Algorithm 1.

Theorem 3.2 *Let Assumptions A and B hold. Then*

$$\lim_{k\to\infty} \frac{\|(B_k - \nabla g_*^2)s_k\|}{\|s_k\|} = 0, \tag{3.34}$$

and $\{x_k\}$ converges superlinearly.

Proof We rewrite (3.23) as

$$\begin{aligned} & \frac{1}{2}\beta\mu_k^2\|H_k - \nabla g_*^{-2}\|_{P^{-1}} \\ \le\ & \|H_k - \nabla g_*^{-2}\|_{P^{-1}} - \|H_{k+1} - \nabla g_*^{-2}\|_{P^{-1}} + M_6\|H_k - \nabla g_*^{-2}\|_{P^{-1}}\nu_k + M_7\nu_k. \end{aligned}$$

Notice that $\{\|H_k - \nabla g_*^{-2}\|_{P^{-1}}\}$ is bounded and $\nu_k = \nu_k(\alpha)$ satisfies (3.15). Summing the above inequalities, we get

$$\frac{1}{2}\beta\sum_{k=0}^{\infty}\mu_k^2\|H_k - \nabla g_*^{-2}\|_{P^{-1}} < \infty.$$

In particular, we have by the definition (3.24) of μ_k

$$\lim_{k\to\infty}\mu_k^2\|H_k - \nabla g_*^{-2}\|_{P^{-1}} = \lim_{k\to\infty}\frac{\|P^{-1}(H_k - \nabla g_*^{-2})y_k\|^2}{\|H_k - \nabla g_*^{-2}\|_{P^{-1}}\|Py_k\|^2} = 0.$$

However, since $\|H_k - \nabla g_*^{-2}\|_{P^{-1}}$ is bounded, it follows that

$$\lim_{k\to\infty}\frac{\|P^{-1}(H_k - \nabla g_*^{-2})y_k\|}{\|Py_k\|} = 0. \tag{3.35}$$

By (2.19) and (2.22), we have

$$\|Py_k\| = \|\nabla g_*^{-1}y_k\| \le \frac{1}{m}\|y_k\| \le \frac{M^2}{m}\|s_k\|.$$

By (2.18) we have

$$\|P^{-1}(H_k - \nabla g_*^{-2})y_k\| = \|\nabla g_*(H_k - \nabla g_*^{-2})y_k\| \ge m\|(H_k - \nabla g_*^{-2})y_k\|.$$

So (3.35) implies

$$\lim_{k\to\infty}\frac{\|(H_k - \nabla g_*^{-2})y_k\|}{\|s_k\|} = 0. \tag{3.36}$$

On the other hand, we have

$$\begin{aligned}
\|(H_k - \nabla g_*^{-2})y_k\| &= \|H_k(\nabla g_*^2 - B_k)\nabla g_*^{-2} y_k\| \\
&\geq \|H_k(\nabla g_*^2 - B_k)s_k\| - \|H_k(\nabla g_*^2 - B_k)(s_k - \nabla g_*^{-2} y_k)\| \\
&= \|H_k(\nabla g_*^2 - B_k)s_k\| - \|H_k(\nabla g_*^2 - B_k)\nabla g_*^{-2}(y_k - \nabla g_*^2 s_k)\| \\
&\geq \|H_k(\nabla g_*^2 - B_k)s_k\| - M_4\nu_k\|H_k(\nabla g_*^2 - B_k)\nabla g_*^{-2}\|\|s_k\| \\
&= \|H_k(\nabla g_*^2 - B_k)s_k\| - o(\|s_k\|),
\end{aligned}$$

where the last inequality follows from (3.20). Notice that$\{\|B_k\|\}$ and $\{\|H_k\|\}$ are bounded, and $\{H_k\}$ is uniformly nonsingular. Therefore, there is a constant $m_3 > 0$ such that $\|H_k(\nabla g_*^2 - B_k)s_k\| \geq m_3\|(\nabla g_*^2 - B_k)s_k\|$ for all k. So we have

$$\|(H_k - \nabla g_*^{-2})y_k\| \geq m_3\|(\nabla g_*^2 - B_k)s_k\| - o(\|s_k\|),$$

and hence (3.36) yields (3.34). In view of Lemma 3.3, the proof is complete. □

Lemma 3.3 and Theorem 3.2 show that when Assumptions A and B are met, (2.13) holds for all k sufficiently large. This means that when k is sufficiently large, Algorithm 1 has a norm descent property.

4 Extension

The convergence results established in the previous sections rely on the assumption that the Jacobian $\nabla g(x)$ is uniformly nonsingular. This is often too restrictive an assumption in practice. The purpose of this section is to modify Algorithm 1 so that the modified algorithm converges globally and superlinearly without the uniform nonsingularity assumption on ∇g. The idea adopted here is similar to that in [12].

Let $s_k = x_{k+1} - x_k = \lambda_k p_k$ and $\bar{y}_k = y_k + \tau_k\|g_k\|s_k$, where $y_k = g(x_k + \delta_k) - g_k$, $\delta_k = g_{k+1} - g_k$, and τ_k is determined by

$$\tau_k = 1 - \frac{1}{\|g_k\|}\min\Big\{\frac{y_k^T s_k}{\|s_k\|^2}, 0\Big\}. \tag{4.1}$$

The modified algorithm then adopts the following update formula in place of (2.14):

$$B_{k+1} = B_k - \frac{B_k s_k s_k^T B_k}{s_k^T B_k s_k} + \frac{\bar{y}_k \bar{y}_k^T}{\bar{y}_k^T s_k}. \tag{4.2}$$

In fact, the only difference between Algorithm 1 and the modified algorithm lies in the different expressions of y_k and $\bar{y}_k$. Note that

$$\begin{aligned}
\bar{y}_k^T s_k &= y_k^T s_k + \tau_k\|g_k\|\,\|s_k\|^2 \\
&= \|g_k\|\,\|s_k\|^2 + y_k^T s_k - \min\{y_k^T s_k, 0\} \\
&\geq \|g_k\|\,\|s_k\|^2.
\end{aligned}$$

Therefore, in the corresponding Step 5 of the modified algorithm, B_k is updated at each iteration.

For the modified algorithm, Lemma 2.1 clearly remains true because it depends only on the line search condition (2.15).

Lemma 4.1 *Let $\{x_k\}$ be generated by the modified algorithm. Then $\{x_k\} \subset \Omega$ and $\{\|g_k\|\}$ converges.*

For the modified algorithm, we make the following assumption instead of Assumption A.

Assumption A′: (i) g is continuously differentiable on an open convex set Ω_1 containing Ω. (ii) The Jacobian of g is symmetric and bounded on Ω_1,

Assumption A′ differs from Assumption A only in that the former does not involve the uniform nonsingularity condition on ∇g. As before, let M be an upper bound of $\|\nabla g(x)\|$ on Ω_1. By the definition (2.16) of Ω, we may assume without loss of generality that $\|g(x)\|$ is bounded on Ω_1. Let L_1 be an upper bound of $\|g(x)\|$ on Ω_1.

It is easily seen that for every k, we have

$$\|\delta_k\| \le M\|s_k\|, \quad \|y_k\| \le M\|\delta_k\| \le M^2\|s_k\|, \tag{4.3}$$

and

$$\begin{aligned}
\|\bar{y}_k\| &\le \|y_k\| + \tau_k\|g_k\|\,\|s_k\| \\
&= \|y_k\| + \Big(1 - \frac{1}{\|g_k\|}\min\Big\{\frac{y_k^T s_k}{\|s_k\|^2}, 0\Big\}\Big)\|g_k\|\,\|s_k\| \\
&\le M^2\|s_k\| + \|g_k\|\,\|s_k\| + \Big|\min\Big\{\frac{y_k^T s_k}{\|s_k\|^2}, 0\Big\}\Big|\,\|s_k\| \\
&\le M^2\|s_k\| + L_1\|s_k\| + \frac{|y_k^T s_k|}{\|s_k\|^2}\|s_k\| \\
&\le (M^2 + L_1 + M^2)\|s_k\| \\
&= C\|s_k\|,
\end{aligned}$$

where $C = 2M^2 + L_1$. The above discussions are summarized in the next lemma.

Lemma 4.2 *Let Assumption A′ hold and $\{x_k\}$ be generated by the modified algorithm. Then there exists a constant $C > 0$ such that the following inequalities hold for all k:*

$$\|\delta_k\| \le M\|s_k\|, \quad \|\bar{y}_k\| \le C\|s_k\|, \tag{4.4}$$

and

$$\|g_k\|\,\|s_k\|^2 \le \bar{y}_k^T s_k \le C\|s_k\|^2. \tag{4.5}$$

Lemmas 4.1 and 4.2 correspond to Lemmas 2.1 and 2.2 for Algorithm 1, respectively. The following lemma then corresponds to Lemma 2.3 for Algorithm 1.

Lemma 4.3 *Let Assumption A′ hold. Suppose that*

$$\liminf_{k\to\infty} \|g_k\| \neq 0. \tag{4.6}$$

Then there is a constant $\zeta > 0$ such that for all k

$$\bar{y}_k^T s_k \geq \zeta \|s_k\|^2. \tag{4.7}$$

If in addition (2.13) holds only for finitely many k's, then we have

$$\sum_{k=0}^{\infty} \|\lambda_k g_k\|^2 < \infty \quad \text{and} \quad \sum_{k=0}^{\infty} \|\lambda_k p_k\|^2 = \sum_{k=0}^{\infty} \|s_k\|^2 < \infty. \tag{4.8}$$

Proof By (4.6), there is a positive constant ζ such that $\|g_k\| > \zeta$ for all k. Therefore, (4.7) follows from (4.5). The inequalities (4.8) are obtained by summing (2.11) directly. □

Noticing (4.5) and (4.7), we can deduce the following lemma from Theorem 2.1 in [2] immediately.

Lemma 4.4 *Let Assumption A′ hold and $\{x_k\}$ be generated by the modified algorithm. If (4.6) holds, then there are positive constants β_j, $j = 1,2,3$ such that the inequalities*

$$\beta_2 \|s_i\|^2 \leq s_i^T B_i s_i \leq \beta_3 \|s_i\|^2 \quad \text{and} \quad \|B_i s_i\| \leq \beta_1 \|s_i\| \tag{4.9}$$

hold for at least a half of indices $i \in \{0, 1, 2, \ldots, k\}$.

Define again the index set J as

$$J = \{i \mid (4.9) \text{ holds}\}.$$

From the remark given just after the proof of Lemma 3.2, we see that Lemma 3.2 remains true for the modified algorithm as long as Assumption A′ holds. We now restate the result as follows.

Lemma 4.5 *Let Assumption A′ hold and $\{x_k\}$ be generated by the modified algorithm. If $\lambda_k \neq 1$, then we have the following estimate for λ_k:*

$$\lambda_k \geq \frac{2(p_k^T B_k p_k - t_k \|p_k\| \|g_k\|) r}{\sigma_1 \|g_k\|^2 + (\sigma_2 + M^2) \|p_k\|^2}, \tag{4.10}$$

where $t_k = \int_0^1 \|\nabla g(x_k + \tau \lambda_k r^{-1} p_k) - \nabla g(x_k + \tau \lambda_{k-1} g_k)\| d\tau$.

So far, we have seen that the modified algorithm inherits those properties of Algorithm 1 which were shown in Lemmas 2.1–2.3 and Lemmas 3.1 and 3.2. With these results, we can prove global convergence of the modified algorithm by following the same steps as those in the proof of Theorem 3.1,

Theorem 4.1 *Let Assumption A′ hold and $\{x_k\}$ be generated by the modified algorithm. Suppose that the sequence $\{x_k\}_{k\in J}$ has an accumulation point $\bar{x}$ where $\nabla g(\bar{x})$ is nonsingular. Then we have*

$$\lim_{k\to\infty} \|g_k\| = 0.$$

Moreover, every accumulation point of $\{x_k\}$ is a solution of (2.1).

Proof Lemma 4.1 ensures that $\{\|g_k\|\}$ converges. So, it suffices to verify

$$\liminf_{k\to\infty} \|g_k\| = 0. \tag{4.11}$$

If (2.13) holds for infinitely many k's, then (4.11) holds obviously. Suppose that (2.13) holds for only finitely many k's. We prove (4.11) by assuming to the contrary that $\|g_k\| \geq \zeta$ for all k with some constant $\zeta > 0$. Let $\{x_k\}_{k\in K}$ be a subsequence of $\{x_k\}_{k\in J}$ converging to $\bar{x}$. Since $\nabla g(\bar{x})$ is nonsingular, it follows that when $k \in K$ is sufficiently large, ∇g_k is uniformly nonsingular. Consequently, there is a positive constant $m' > 0$ such that $\|G_k^{-1}\| \leq m'$ holds for all $k \in K$ large enough, where G_k is defined by (2.23). We consider the corresponding steplength sequence $\{\lambda_k\}_{k\in K}$. If $\lambda_k \neq 1$, then (4.10) yields

$$\begin{aligned}
\lambda_k &\geq \frac{2(p_k^T B_k p_k - t_k\|p_k\|\|g_k\|)r}{\sigma_1\|g_k\|^2 + (\sigma_2 + M^2)\|p_k\|^2} \\
&= \frac{2(p_k^T B_k p_k - t_k\|p_k\|\|G_k^{-1}B_k p_k\|)r}{\sigma_1\|G_k^{-1}B_k p_k\|^2 + (\sigma_2 + M^2)\|p_k\|^2} \\
&\geq \frac{2(p_k^T B_k p_k - t_k m'\|p_k\|\|B_k p_k\|)r}{\sigma_1 m'\|B_k p_k\|^2 + (\sigma_2 + M^2)\|p_k\|^2} \\
&\geq \frac{2(\beta_2\|p_k\|^2 - m'\beta_1 t_k\|p_k\|^2)r}{m'\sigma_1\beta_1^2\|p_k\|^2 + (\sigma_2 + M^2)\|p_k\|^2} \\
&= \frac{2(\beta_2 - m'\beta_1 t_k)r}{m'\sigma_1\beta_1^2 + (\sigma_2 + M^2)},
\end{aligned}$$

where the last inequality follows from (4.9). Lemmas 4.1 and 4.2 imply that $\{\|g_k\|\}$ converges, $\lambda_k \to 0$, $\|\lambda_k p_k\| \to 0$, and hence $t_k \to 0$. Therefore we deduce that $\{\lambda_k\}_{k\in K}$ is bounded away from zero. This together with (4.8) induces a contradiction to the assumption that $\|g_k\| \geq \zeta$ for all k. The proof is then complete. □

To establish the superlinear convergence of the modified algorithm, we need the following additional assumptions.

Assumption B′ (i) $x_k \to x^*$, $g_* = 0$ and ∇g_* is nonsingular.

(ii) ∇g is Hölder continuous at x^*.

It is easy to see that under the condition (i) in Assumption B′, there is a positive constant m_3 and a neighbourhood $U(x^*)$ of x^* such that $\|\nabla g(x)p\| \geq m_3\|p\|$ for all $x \in U(x^*)$ and $p \in R^n$. Taking this into account, we have the following result.

Lemma 4.6 *Let Assumptions A′ and B′ hold. Then the following statements hold:*

(i) For all k sufficiently large, $\tau_k = 1$ and $\bar{y}_k = y_k + \|g_k\| s_k$.

(ii) There are positive constants $\beta_i, i = 1,2,3$ such that (4.9) hold for at least a half of indices $i \in \{0,1,2,\ldots,k\}$.

(iii) There is a positive constant $\tilde{\lambda}'$ such that $\lambda_k \geq \tilde{\lambda}'$ holds for all $k \in J$ sufficiently large.

Proof (i) First it is easy to see that for every k,

$$\begin{aligned} y_k^T s_k &= s_k^T [g(x_k + \delta_k) - g(x_k)] \\ &= s_k^T \int_0^1 \nabla g(x_k + \tau\delta_k) d\tau \delta_k \\ &= s_k^T \int_0^1 \nabla g(x_k + \tau\delta_k) d\tau \int_0^1 \nabla g(x_k + \tau s_k) d\tau s_k \\ &= \|\nabla g_* s_k\|^2 + o(\|s_k\|^2) \\ &\geq m_3^2 \|s_k\|^2 + o(\|s_k\|^2). \end{aligned} \tag{4.12}$$

This particularly implies that $y_k^T s_k > \frac{m_3^2}{2}\|s_k\|^2 > 0$ for all k sufficiently large. It then follows from (4.1) that $\tau_k = 1$ holds for all k sufficiently large and thus $\bar{y}_k = y_k + \|g_k\| s_k$.

(ii) Combining (i) with (4.3) yields

$$\|\bar{y}_k\| \leq \|y_k\| + \|g_k\| \, \|s_k\| \leq (M^2 + \|g_k\|)\|s_k\|.$$

Therefore we conclude from Theorem 2.1 in [2] that there are positive constants $\beta_i, i = 1,2,3$ such that (4.9) holds for at least a half of indices $i \in \{0,1,2,\ldots,k\}$.

(iii) Notice again that the condition (i) in Assumption B′ ensures that there exists a constant $m_4 > 0$ such that when k is sufficiently large, G_k^{-1} exists and $\|G_k^{-1}\| \leq m_4$. Therefore, following the proof of Theorem 4.1, we can verify the conclusion of the lemma. □

Since ∇g_k is uniformly nonsingular for all k sufficiently large, it is not difficult to see that under Assumptions A′ and B′, the conditions in Assumptions A and B hold in a neighbourhood of x^*. So, for the modified algorithm, we can establish results similar to Lemmas 3.3, Lemmas 3.4 and 3.5. We summarize these results as the following lemma.

Lemma 4.7 *Let Assumptions A′ and B′ hold and $\{x_k\}$ be generated by the modified algorithm. Then the following statements hold:*

(i) The Dennis-Moré condition

$$\lim_{k\to\infty} \frac{\|(B_k - \nabla g_*^2) p_k\|}{\|p_k\|} = 0 \tag{4.13}$$

ensures that $\lambda_k \equiv 1$ for all k sufficiently large. Moreover, if (4.13) holds, then $\{x_k\}$ converges superlinearly.

(ii) For any fixed $\gamma > 0$, we have

$$\sum_{k=0}^{\infty} \|x_k - x^*\|^{\gamma} < \infty. \tag{4.14}$$

Moreover, we have

$$\sum_{k=0}^{\infty} \nu_k(\gamma) < \infty, \tag{4.15}$$

where $\nu_k(\gamma) = \max\{\|x_k - x^*\|^{\gamma}, \|x_{k+1} - x^*\|^{\gamma}\}$.

(iii) Let $\nu_k = \nu_k(\alpha)$, *where* α *is the modulus of Hölder continuity of* ∇g. *Then there exist positive constants* $\bar{M}$ *and* $\bar{m}$ *such that for all* k *sufficiently large*

$$\|\bar{y}_k - \nabla g_*^2 s_k\| \le [\bar{M}\nu_k + \|g_k\|\,]\,\|s_k\| \quad \text{and} \quad \|\bar{y}_k\| \ge \bar{m}\|s_k\|. \tag{4.16}$$

Proof Part (i) follows from Lemma 3.3 directly since the proof of Lemma 3.3 is independent of the update formula. Moreover, the proof of Theorem 4.1 indicates that $\lambda_k \ge \tilde{\lambda}$ for all $k \in J$ with some constant $\tilde{\lambda} > 0$. This means that for every $i \in J$, we have either

$$\|g_{i+1}\|^2 \le \rho^2 \|g_i\|^2$$

or

$$\|g_{i+1}\|^2 \le (1 - \sigma_1 \lambda_i^2 + \epsilon_i)\|g_i\|^2 \le (1 - \sigma_1 \tilde{\lambda}^2 + \epsilon_i)\|g_i\|^2.$$

So, we can obtain (ii) in a way similar to the proof for Lemma 3.4. Now we prove (iii). Note that, by Lemma 4.6, $\tau_k = 1$ holds for all k sufficiently large. Thus we get

$$\begin{aligned}
\|\bar{y}_k - \nabla g_*^2 s_k\| &= \Big\| y_k - \nabla g_*^2 s_k + \|g_k\| s_k \Big\| \\
&\le \|y_k - \nabla g_*^2 s_k\| + \|g_k\|\,\|s_k\| \\
&= \Big\| \int_0^1 \nabla g(x_k + \tau\delta_k)d\tau \int_0^1 \nabla g(x_k + \tau s_k)d\tau s_k - \nabla g_*^2 s_k \Big\| + \|g_k\|\,\|s_k\| \\
&\le \Big[\Big\| \int_0^1 (\nabla g(x_k + \tau\delta_k) - \nabla g_*)d\tau \int_0^1 \nabla g(x_k + \tau s_k)d\tau \Big\| \\
&\quad + \Big\| \nabla g_* \int_0^1 \nabla (g(x_k + \tau s_k) - \nabla g_*)d\tau \Big\| \Big] \|s_k\| + \|g_k\|\,\|s_k\| \\
&\le \Big[\int_0^1 \|\nabla g(x_k + \tau\delta_k) - \nabla g_*\|d\tau \int_0^1 \|\nabla g(x_k + \tau s_k)\|d\tau \\
&\quad + \|\nabla g_*\| \int_0^1 \|\nabla g(x_k + \tau s_k) - \nabla g_*\|d\tau \Big] \|s_k\| + \|g_k\|\,\|s_k\| \\
&\le \Big[M M_3 \int_0^1 \|x_k - x^* + \tau\delta_k\|^{\alpha} d\tau + M M_3 \int_0^1 \|x_k - x^* + \tau s_k\|^{\alpha} d\tau \Big] \|s_k\| \\
&\quad + \|g_k|\,\|s_k\| \\
&\le M M_3 \Big[(\|x_k - x^*\| + \|\delta_k\|)^{\alpha}
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^1 [(1-\tau)\|x_k - x^*\| + \tau\|x_{k+1} - x^*\|]^\alpha \Big] d\tau \|s_k\| + \|g_k\|\,\|s_k\| \\
\leq\ & MM_3\Big[(\|x_k - x^*\| + M\|s_k\|)^\alpha + \nu_k\Big]\|s_k\| + \|g_k\|\,\|s_k\| \\
\leq\ & MM_3\Big\{\Big[\|x_k - x^*\| + M(\|x_k - x^*\| + \|x_{k+1} - x^*\|)\Big]^\alpha + \nu_k\Big\}\|s_k\| \\
& +\|g_k\|\,\|s_k\| \\
\leq\ & MM_3\{[(2M+1)^\alpha + 1]\nu_k + \|g_k\|\}\|s_k\|,
\end{aligned}
$$

where M_3 is the Hölder constant given by (3.19). Putting $\bar{M} = MM_3[(2M+1)^\alpha + 1]$, we get the first inequality of (4.16). Moreover, we have

$$
\begin{aligned}
\|\bar{y}_k\| &= \Big\|y_k + \|g_k\|s_k\Big\| \\
&\geq \|y_k\| - \|g_k\|\,\|s_k\| \\
&= \Big\|\int_0^1 \nabla g(x_k + \tau\delta_k)d\tau \int_0^1 \nabla g(x_k + \tau s_k)d\tau s_k\Big\| - \|g_k\|\,\|s_k\| \\
&= \|\nabla g_*^2 s_k\| + o(\|s_k\|),
\end{aligned}
$$

where the last equality follows from Assumption B′. Since ∇g_* is nonsingular, this yields the second inequality of (4.16). □

In a similar way to Lemma 3.6 in [12], we can prove the following lemma for the modified algorithm, which corresponds to Lemma 3.6 for Algorithm 1. For completeness, we give a proof.

Lemma 4.8 *Let Assumptions A′ and B′ hold and B_k be updated by (4.2). Denote $P = \nabla g_*^{-1}$, $B_k^{-1} = H_k$ and $B_{k+1}^{-1} = H_{k+1}$. Then there are positive constants $\alpha_i, i = 1,2,\cdots,7$ and $\beta \in (0,1)$ such that for all k sufficiently large*

$$
\|B_{k+1} - \nabla g_*^2\|_P \leq \|B_k - \nabla g_*^2\|_P + \alpha_1\nu_k + \alpha_2\|g_k\| \tag{4.17}
$$

and

$$
\begin{aligned}
&\|H_{k+1} - \nabla g_*^{-2}\|_{P^{-1}} \\
\leq\ & (\sqrt{1 - \beta\bar{\mu}_k^2} + \alpha_3\nu_k + \alpha_4\|g_k\|)\|H_k - \nabla g_*^{-1}\|_{P^{-1}} + \alpha_5\nu_k + \alpha_6\|g_k\|,
\end{aligned} \tag{4.18}
$$

where $\nu_k = \max\{\|x_k - x^*\|^\alpha, \|x_{k+1} - x^*\|^\alpha\}$ and $\bar{\mu}_k$ is given by

$$
\bar{\mu}_k = \frac{\|P^{-1}[H_k - \nabla g_*^{-2}]\bar{y}_k\|}{\|H_k - \nabla g_*^{-2}\|_{P^{-1}}\|P\bar{y}_k\|}.
$$

In particular, $\{\|B_k\|\}$ and $\{\|H_k\|\}$ are bounded.

Proof From the update formula (4.2), we have

$$
\|B_{k+1} - \nabla g_*^2\|_P
$$

$$
\begin{aligned}
&= \Big\| B_k - \nabla g_*^2 - \frac{B_k s_k s_k^T B_k}{s_k^T B_k s_k} + \frac{\bar{y}_k \bar{y}_k^T}{\bar{y}_k^T s_k} \Big\|_P \\
&\le \Big\| B_k - \nabla g_*^2 - \frac{B_k s_k s_k^T B_k}{s_k^T B_k s_k} + \frac{y_k y_k^T}{y_k^T s_k} \Big\|_P + \Big\| \frac{\bar{y}_k \bar{y}_k^T}{\bar{y}_k^T s_k} - \frac{y_k y_k^T}{y_k^T s_k} \Big\|_P \\
&\le \|B_k - \nabla g_*^2\|_P + M_5 \nu_k + \Big\| \frac{\bar{y}_k \bar{y}_k^T}{\bar{y}_k^T s_k} - \frac{y_k y_k^T}{y_k^T s_k} \Big\|_P , \qquad (4.19)
\end{aligned}
$$

where the last inequality follows from (3.22). Moreover, by Lemma 4.6, for all k large enough, we have $\bar{y}_k = y_k + \|g_k\| s_k$, and hence

$$
\begin{aligned}
&\Big\| \frac{\bar{y}_k \bar{y}_k^T}{\bar{y}_k^T s_k} - \frac{y_k y_k^T}{y_k^T s_k} \Big\|_P \\
&= \Big\| \frac{(y_k + \|g_k\| s_k)(y_k + \|g_k\| s_k)^T}{(y_k + \|g_k\| s_k)^T s_k} - \frac{y_k y_k^T}{y_k^T s_k} \Big\|_P \\
&= \Big\| \frac{y_k^T s_k (y_k + \|g_k\| s_k)(y_k + \|g_k\| s_k)^T - (y_k + \|g_k\| s_k)^T s_k y_k y_k^T}{(y_k + \|g_k\| s_k)^T s_k (y_k^T s_k)} \Big\|_P \\
&= \Big\| \frac{y_k^T s_k (y_k s_k^T + s_k y_k^T + \|g_k\| s_k s_k^T) - \|s_k\|^2 y_k y_k^T}{(y_k^T s_k + \|g_k\| \|s_k\|^2)(y_k^T s_k)} \Big\|_P \|g_k\| \\
&\le \Big\| \frac{y_k^T s_k (y_k s_k^T + s_k y_k^T + \|g_k\| s_k s_k^T) - \|s_k\|^2 y_k y_k^T}{(y_k^T s_k + o(\|s_k\|^2))(y_k^T s_k)} \Big\|_F \|P\|_F^2 \|g_k\| \\
&\le \frac{|y_k^T s_k| (\|y_k s_k^T\|_F + \|s_k y_k^T\|_F + \|g_k\| \|s_k s_k^T\|_F) + \|s_k\|^2 \|y_k y_k^T\|_F}{(\|\nabla g_* s_k\|^2 + o(\|s_k\|^2))(\|\nabla g_* s_k\|^2 + o(\|s_k\|^2))} \|P\|_F^2 \|g_k\| \\
&\le \frac{\|y_k\| \|s_k\| (\|y_k\| \|s_k\| + \|s_k\| \|y_k\| + \|g_k\| \|s_k\|^2) + \|s_k\|^2 \|y_k\|^2}{(\|\nabla g_* s_k\|^2 + o(\|s_k\|^2))(\|\nabla g_* s_k\|^2 + o(\|s_k\|^2))} \|P\|_F^2 \|g_k\| \\
&= \frac{3\|y_k\|^2 \|s_k\|^2 + \|g_k\| \|y_k\| \|s_k\|^3}{(\|\nabla g_* s_k\|^2 + o(\|s_k\|^2))(\|\nabla g_* s_k\|^2 + o(\|s_k\|^2))} \|P\|_F^2 \|g_k\| \\
&\le \frac{3M^4 \|s_k\|^4 + M^2 \|g_k\| \|s_k\|^4}{(\|\nabla g_* s_k\|^2 + o(\|s_k\|^2))(\|\nabla g_* s_k\|^2 + o(\|s_k\|^2))} \|P\|_F^2 \|g_k\| \\
&= \frac{3M^4 \|s_k\|^4 + o(\|s_k\|^4)}{(\|\nabla g_* s_k\|^2 + o(\|s_k\|^2))(\|\nabla g_* s_k\|^2 + o(\|s_k\|^2))} \|P\|_F^2 \|g_k\|, \qquad (4.20)
\end{aligned}
$$

where the second and last inequalities follow from (4.12) and (4.3), respectively. Since ∇g_* is nonsingular, we get (4.17) from (4.19) and (4.20).

Now we prove (4.18). First, we have

$$
\begin{aligned}
\|P\bar{y}_k - P^{-1} s_k\| &\le \|P\| \|\bar{y}_k - P^{-2} s_k\| \\
&= \|P\| \|\bar{y}_k - \nabla g_*^2 s_k\| \\
&\le \|P\| (\bar{M}\nu_k + \|g_k\|) \|s_k\| \\
&\le \bar{m}^{-1} \|P\| (\bar{M}\nu_k + \|g_k\|) \|\bar{y}_k\|, \qquad (4.21)
\end{aligned}
$$

where the second and the last inequalities are obtained from (4.16). Similarly we have

$$
\|\nabla g_*^{-2} \bar{y}_k - s_k\| \le \|P\|^2 \|\bar{y}_k - \nabla g_*^2 s_k\| \le \bar{m}^{-1} \|P\|^2 (\bar{M}\nu_k + \|g_k\|) \|\bar{y}_k\|. \qquad (4.22)
$$

Notice again that the only difference between (4.2) and (2.14) lies in the expressions of $\bar{y}_k$ and y_k. Since $\nu_k \to 0$ by (4.15) and $\|g_k\| \to 0$, we obtain the following inequality by using $\bar{y}_k$ instead of y_k in (3.33):

$$\begin{aligned}\|H_{k+1} - \nabla g_*^{-2}\|_{P^{-1}} \quad \leq \quad & \Big(\sqrt{1-\beta\mu_k^2} + \alpha_1' \frac{\|P^{-1}s_k - P\bar{y}_k\|}{\|P\bar{y}_k\|}\Big)\|H_k - \nabla g_*^2\|_{P^{-1}} \\ & + \alpha_2' \frac{\|s_k - \nabla g_*^{-2}\bar{y}_k\|}{\|P\bar{y}_k\|},\end{aligned}$$

where $\alpha_1' > 0$, $\alpha_2' > 0$ and $\beta \in (0,1)$ are some constants. Combining this inequality with (4.16), (4.21) and (4.22), we get (4.18).

Finally, by Lemma 4.7, we have $\sum_{k=0}^{\infty} \nu_k < \infty$ and $\sum_{k=0}^{\infty} \|g_k\| \leq M \sum_{k=0}^{\infty} \|x_k - x^*\| < \infty$. Hence (4.17) and (4.18) together with Lemma 3.3 in [4] indicate that both $\{\|B_k - \nabla g_*^2\|_P\}$ and $\{\|H_k - \nabla g_*^{-2}\|_{P^{-1}}\}$ converge. In particular, $\{\|B_k\|\}$ and $\{\|H_k\|\}$ are bounded. □

To establish a superlinear convergence result like Theorem 3.2, we notice that $\sum_{k=0}^{\infty} \|g_k\| \leq M \sum_{k=0}^{\infty} \|x_k - x^*\| < \infty$. This means that in (4.17) and (4.18), $\|g_k\|$ as well as ν_k plays the same role as ν_k does in (3.22) and (3.23). Therefore, following the same steps as those in the proof of Theorem 3.2, we can show the following result.

Theorem 4.2 *Let Assumptions A' and B' hold. Then $\{x_k\}$ generated by the modified algorithm converges superlinearly.*

We conclude the paper with some remarks on the nonsingularity of ∇g_*, since it is necessary for both global and superlinear convergence for the modified algorithm. First, we point out that the conditions used in the paper are weaker than those used in [10] which establishes global convergence of Boryden's method by assuming that the Jacobian matrices are uniformly nonsingular and Lipschitz continuous.

For the unconstrained optimization problem (2.2), the nonsingularity of $\nabla g(x^*) = \nabla^2 f(x^*)$ is a weaker requirement. It holds especially when ∇g_* is positive definite, which is a sufficient condition for x^* to be a strict local minimum of the function f. For the equality constrained optimization problem (2.3), let the function g be given by (2.4). Then ∇g_* is nonsingular whenever the columns of $\nabla h(z^*)$ are linearly independent and $p^T \nabla_z L(x^*)p > 0$ holds for every $p \in \{p \,|\, \nabla h(z^*)^T p = 0\}$, where L is defined by

$$L(x) = L(z,v) = \nabla f(z) + \nabla h(z)v.$$

In other words, the second order sufficiency conditions along with the linear independence constraint qualification yield the nonsingularity of ∇g_*. There are some other practical problems such as the discretized two-point boundary value problem and the elliptic boundary value problem where ∇g is generally nonsingular. Indeed, for most of these problems, ∇g exhibits irreducibly diagonal dominance property, which ensures its nonsingularity. For the detail, see Chapter 1 in [13].

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