QUEUE LENGTH DISTRIBUTION IN A FIFO SINGLE-SERVER QUEUE WITH MULTIPLE ARRIVAL STREAMS HAVING DIFFERENT SERVICE TIME DISTRIBUTIONS

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Abstract

This paper considers the queue length distribution in a class of FIFO single-server queues with (possibly correlated) multiple arrival streams, where the service time distribution of customers from each arrival stream may differ from one another among streams. It is widely recognized that the queue length distribution in a FIFO queue with multiple non-Poissonian arrival streams having different service time distributions is very hard to analyze, since we have to keep track of the complete order of customers in the queue to describe the queue length dynamics. In this paper, we provide an alternative way to solve the problem for a class of such queues. We characterize the stationary joint queue length distribution in terms of the joint probability generating function, by considering the joint distribution of the number of customers arriving from each stream during the stationary attained waiting time. Further we provide recursion formulas to compute the stationary joint queue length distribution, the stationary distribution of the total number of customers and the marginal distribution of the number of customers from a particular stream.

Subject Classification: Queue, Algorithm: queue length distribution in a FIFO queue with multiple arrival streams having different service time distributions.

This paper considers the queue length distribution in work-conserving FIFO single-server queues with multiple arrival streams. An analytically tractable model of such a queue is $\sum M_i/GI_i/1$ queue, i.e., all arrival streams follow independent Poisson processes and the service time distribution of customers from each arrival stream are i.i.d. (independent and identically distributed) which may differ from one another among streams. In this model, the dynamics of the total number of customers can be formulated as a $M/GI/1$ queue. Let P denote the number of arrival streams and $H_k^*(s)$ $(k = 1, \ldots, P)$ denote the LST (Laplace-Stieltjes transform) of the service time distribution of customers from the kth stream with arrival rate λ_k. Then the probability generating function of the stationary distribution of the total number Y of customers in the system is given by (Wollf 1989)

$$E[z^Y] = \frac{(1-\rho)(z-1)\sum_{k=1}^{P}\frac{\lambda_k}{\lambda}H_k^*(\lambda-\lambda z)}{z-\sum_{k=1}^{P}\frac{\lambda_k}{\lambda}H_k^*(\lambda-\lambda z)}, \qquad |z| \le 1,$$

where ρ denotes the overall utilization factor and $\lambda = \sum_{k=1}^{P}\lambda_k$. Further the probability generating function of the stationary joint distribution of queue lengths Y_k $(k = 1, \ldots, P)$ of customers from stream k is found to be

$$E\left[z_1^{Y_1}\cdots z_P^{Y_P}\right] = \frac{(1-\rho)\left(\sum_{k=1}^{P}\frac{\lambda_k}{\lambda}z_k - 1\right)\sum_{k=1}^{P}\frac{\lambda_k}{\lambda}H_k^*\left(\lambda-\sum_{k=1}^{P}\lambda_k z_k\right)}{\sum_{k=1}^{P}\frac{\lambda_k}{\lambda}z_k - \sum_{k=1}^{P}\frac{\lambda_k}{\lambda}H_k^*\left(\lambda-\sum_{k=1}^{P}\lambda_k z_k\right)}, \qquad |z_k| \le 1 \ (k = 1, \ldots, P).$$

Note also that the probability generating function of the stationary joint queue length distribution satisfies (Keilson and Servi 1988)

$$E\left[z_1^{Y_1}\cdots z_P^{Y_P}\right] = W^*\left(\lambda-\sum_{k=1}^{P}\lambda_k z_k\right)\sum_{k=1}^{P}\frac{\lambda_k}{\lambda}H_k^*\left(\lambda-\sum_{k=1}^{P}\lambda_k z_k\right), \qquad |z_k| \le 1 \ (k = 1, \ldots, P),$$

where $W^*(s)$ denotes the LST of the stationary waiting time distribution (which is identical among all streams due to the PASTA property (Wolff 1992)):

$$W^*(s) = \frac{(1-\rho)s}{s-\lambda+\sum_{k=1}^{P}\lambda_k H_k^*(s)}, \qquad Re(s) > 0.$$

Note that those simple analytical results come mainly from three facts: (i) Poisson process is closed under superposition, i.e., the superposition of independent Poisson processes with rates λ_k $(k = 1, \ldots, P)$ is a Poisson process with rate λ, (ii) the superposed arrival process and service times are independent,

i,e, given an arrival, the probability that this customer is from stream k is equal to λ_k/λ, independent of the superposed arrival process, and (iii) the PASTA property (Wollf 1982).

Now we consider a FIFO single-server queue with multiple non-Poissonian arrival streams. If the service times of customers from all streams are i.i.d., the problem is reduced to characterize the superposed arrival process, and the resulting queue is formulated as a $G/GI/1$ queue in general. A certain class of arrival processes such as MAP (Markovian arrival process) introduced by Lucantoni, Meier-Hellstern and Neuts (1990) is closed under superposition, i.e., the superposition of independent MAP's can be represented by MAP. Thus for such a class of arrival processes, the waiting time distribution and the distribution of the total number of customers can be analyzed in the same way as the queue with a single arrival stream with i.i.d. service times.

Contrary, when the service time distributions of multiple non-Poissonian streams differ from one another, the analysis of such queues is not so easy, since the superposed arrival process and service times are *dependent*. For example, let us consider a FIFO single-server queue with two independent arrival streams, one of which is a Poisson arrival stream with i.i.d. exponential service times with mean μ_1^{-1} and the other is a two-stage Erlang arrival stream with i.i.d. exponential service times with mean μ_2^{-1}, where $\mu_1 \neq \mu_2$. Even though the system dynamics can be formulated as a continuos-time Markov chain by keeping track of the type of customers (i.e., from which stream each customer arrives) in the waiting line and the service position, the structure of this Markov chain is terribly complicated and therefore it is formidable to obtain the stationary distribution from this formulation. Thus an analytical treatment in full generality to characterize queues with multiple non-Poissonian arrival streams having different service time distributions had not been reported for a long time until Takine and Hasegawa (1994a) showed simple formulas for work in system in a $MAP/G/1$ queue with state-dependent service times (see (7) below). Thus, with the result in Takine and Hasegawa (1994a) and Little's law, we can obtain the mean queue length. To the best of our knowledge, however, no analytical results on higher moments of the queue length or the queue length distribution itself in the $MAP/G/1$ queue with state-dependent service times have not been available so far. Related studies on work in system (or waiting time) in queues with multiple non-Poissonian arrival streams having different service time distributions can be found in Asmussen (1991), Machihara (1993), Takine (1996b) and Zhu and Prabhu (1991). Regterschot and de Smit (1986) consider the $MMPP/G/1$ queue with state-dependent service times and derives analytical expressions for the LST of the waiting time distribution and the probability generating function of the distribution of the total number of customers

by a matrix factorization method which heavily relies on complex plain methods. Note also that the queue length and the waiting time distributions in such a queue under priority disciplines has been analyzed by Sugahara et al. (1995), Takine (1996a) and Takine (1998).

In this paper, we consider the queue length distribution in a work-conserving FIFO single-server queue with (possibly correlated) multiple non-Poissonian arrival streams, where the service time distribution of customers from each arrival stream may differ from one another among streams. In particular, we assume that the arrival process is a marked MAP with representation $(\boldsymbol{C}, \boldsymbol{D}_1, \ldots, \boldsymbol{D}_P)$, where $\boldsymbol{C}$ and $\boldsymbol{D}_k$ ($k = 1, \ldots, P$) are $M \times M$ matrices whose (i,j)th elements are denoted by $C_{i,j}$ and $D_{k,i,j}$, respectively (He 1996). In the marked MAP, the infinitesimal generator of the underlying continuous-time Markov chain which governs customer arrivals is given by $\boldsymbol{C} + \boldsymbol{D}_1 + \cdots + \boldsymbol{D}_P$, where $\boldsymbol{C}$ has negative diagonal elements, and off-diagonal elements in $\boldsymbol{C}$ and all elements in $\boldsymbol{D}_k$ ($k = 1, \ldots, P$) are non-negative. Note that

$$(\boldsymbol{C} + \boldsymbol{D})\, \boldsymbol{e} = \boldsymbol{0}, \tag{1}$$

where

$$\boldsymbol{D} = \sum_{k=1}^{P} \boldsymbol{D}_k,$$

and $\boldsymbol{e}$ denotes an $M \times 1$ vector whose elements are all equal to one. We assume that the Markov chain represented by the infinitesimal generator $\boldsymbol{C} + \boldsymbol{D}$ is positive recurrent. When a state transition driven by $\boldsymbol{D}_k$ ($k = 1, \ldots, P$) occurs, a customer from stream k arrives to the system. Hereafter we call customers from stream k ($k = 1, \ldots, P$) class k customers. Let $N_{(x,y]}(k)$ denote the number of class k customers arrivals during an interval $(x, y]$. Then the (i,j)th element of

$$\exp\left[\left(\boldsymbol{C} + \sum_{k=1}^{P} z_k \boldsymbol{D}_k\right) x\right]$$

represents

$$E\left[z_1^{N_{(\tau,\tau+x]}(1)} \cdots z_P^{N_{(\tau,\tau+x]}(P)} 1_{\{S_{\tau+x}=j\}} \mid S_\tau = i\right],$$

where S_t denotes the state of the underlying Markov chain at time t and $1_{\{\chi\}}$ denotes the indicator function of event χ. Note that when $P = 1$, marked MAP is reduced to MAP. For the details of the marked MAP, readers are referred to He (1996). Throughout the paper, matrices and vectors are denoted in bold capital letters and bold small letters, respectively.

We assume that service times of class k ($k = 1, \ldots, P$) customers who arrive with the transition from state i to state j are i.i.d. according to a distribution function $H_{k,i,j}(x)$ with mean $h_{k,i,j}$. We define

$\boldsymbol{D}_k(x)$ ($k = 1, \ldots, P$) as an $M \times M$ matrix whose (i,j)th element $D_{k,i,j}(x)$ is given by

$$D_{k,i,j}(x) = D_{k,i,j} H_{k,i,j}(x), \qquad k = 1, \ldots, P,\ i, j = 1, \ldots, M.$$

Let $\boldsymbol{D}_k^*(s)$ ($k = 1, \ldots, P$) denote the LST of $\boldsymbol{D}_k(x)$:

$$\boldsymbol{D}_k^*(s) = \int_{0-}^{\infty} e^{-sx} d\boldsymbol{D}_k(x), \qquad Re(s) > 0, k = 1, \ldots, P.$$

At a glance, the service time structure for each arrival stream might look unnecessarily complicated. However, the above dependency between an arrival process and service times naturally arises in the superposition of independent MAP streams having different service time distributions. Readers are referred to Takine and Hasegawa (1994a) which gives a simple example of such a case. Note also that MAP is weakly dense in simple stationary point processes (Asmussen and Koole 1993). Thus, approximately with any desired accuracy, our formulation is capable for representing *any* FIFO single-server queue with multiple stationary arrival streams, each of which has i.i.d. service times whose distribution may differ from one another among streams.

For simplicity in description, let $\boldsymbol{D}(x)$ be

$$\boldsymbol{D}(x) = \sum_{k=1}^{P} \boldsymbol{D}_k(x), \tag{2}$$

and the LST of $\boldsymbol{D}(x)$ is denoted by $\boldsymbol{D}^*(s)$:

$$\boldsymbol{D}^*(s) = \int_{0-}^{\infty} e^{-sx} d\boldsymbol{D}(x) = \sum_{k=1}^{P} \boldsymbol{D}_k^*(s), \qquad Re(s) > 0. \tag{3}$$

We denote by $\boldsymbol{\pi}$ the stationary probability vector of the underlying Markov chain and it satisfies

$$\boldsymbol{\pi}(\boldsymbol{C} + \boldsymbol{D}) = \boldsymbol{0}, \qquad \boldsymbol{\pi}\boldsymbol{e} = 1. \tag{4}$$

Then the mean arrival rate λ_k ($k = 1, \ldots, P$) of class k customers is given by

$$\lambda_k = \boldsymbol{\pi}\boldsymbol{D}_k\boldsymbol{e}, \qquad k = 1, \ldots, P.$$

Let ρ_k denote the utilization factor of class k customers, i.e.,

$$\rho_k = \boldsymbol{\pi} \int_{0-}^{\infty} x d\boldsymbol{D}_k(x) \boldsymbol{e}, \qquad k = 1, \ldots, P.$$

Furthermore, we denote the overall arrival rate by $\lambda = \sum_{k=1}^{P} \lambda_k$ and the overall utilization factor by $\rho = \sum_{k=1}^{P} \rho_k$. In the rest of the paper, we assume $\rho < 1$, which ensure that all customers arriving to the system are eventually served.

We shall provide some preliminary results given in Takine and Hasegawa (1994a), which analyzed the workload process in the $MAP/G/1$ queue with state-dependent service times. As explained in Takine and Hasegawa (1994a), the arrival process considered in this paper is equivalent to a MAP with state-dependent service times. Therefore, in the rest of this section, we summarize the results in Takine and Hasegawa (1994a), which will be used later in the paper.

We define $\boldsymbol{Q}$ as an $M \times M$ matrix which represents the infinitesimal generator of an underlying Markov chain obtained by excising the busy periods. Namely, if we observe the system only when the server is idle, the observed underlying Markov chain is governed by the generator $\boldsymbol{Q}$. The matrix $\boldsymbol{Q}$ satisfies

$$\boldsymbol{Q} = \boldsymbol{C} + \int_{0-}^{\infty} d\boldsymbol{D}(x) \exp[\boldsymbol{Q}x]. \tag{5}$$

Note here that $\exp[\boldsymbol{Q}x]$ denotes the transition probability matrix of the underlying Markov chain during the first passage time to the idle state of the server given that the first passage time starts with the amount x of the initial work. Let $\boldsymbol{\kappa}$ denote a $1 \times M$ vector which satisfies

$$\boldsymbol{\kappa Q} = \boldsymbol{0}, \qquad \boldsymbol{\kappa e} = 1. \tag{6}$$

Let V denote a generic random variable representing the amount of stationary work in system (i.e., the sum of the unfinished work of all customers in the steady state). Also let S denote a generic random variable representing the state of the underlying Markov chain in the steady state. With those, we define $\boldsymbol{v}(x)$ as a $1 \times M$ vector whose jth element $v_j(x)$ represents the distribution function of the amount of stationary work in system when the underlying Markov chain is in state j:

$$v_j(x) = \Pr(V \le x, S = j), \qquad j = 1, \ldots, M.$$

Furthermore, let $\boldsymbol{v}^*(s)$ denote a $1 \times M$ vector whose jth element represents the Laplace-Stieltjes transform (LST) of $v_j(x)$. We then have (Takine and Hasegawa 1994a)

$$\boldsymbol{v}^*(s) = (1-\rho)s\boldsymbol{\kappa}\left[s\boldsymbol{I} + \boldsymbol{C} + \boldsymbol{D}^*(s)\right]^{-1}, \qquad Re(s) > 0, \tag{7}$$

from which moments of the distribution of V can be recursively computed. In particular, the average of stationary work in system is given by

$$E[V] = \frac{1}{1-\rho}\left[\boldsymbol{\pi D}^{(2)}\boldsymbol{e} + \left(\boldsymbol{\pi D}^{(1)} + (1-\rho)\boldsymbol{\kappa}\right)(\boldsymbol{e\pi} - \boldsymbol{C} - \boldsymbol{D})^{-1}\boldsymbol{D}^{(1)}\boldsymbol{e} - \rho\right],$$

where

$$\boldsymbol{D}^{(n)} = \lim_{s \to 0+} \frac{(-1)^n}{n!} \frac{d^n}{ds^n} \boldsymbol{D}^*(s) \quad (n = 1, 2).$$

Let $\boldsymbol{w}_k(x)$ (resp. $\boldsymbol{r}_k(x)$) denote a $1 \times M$ vector whose jth element $w_{k,j}(x)$ (resp. $r_{k,j}(x)$) represents the distribution function of the actual waiting time (resp. sojourn time) of class k customers when the underlying Markov chain is in state j immediately after their arrivals, i.e., for $k = 1, \ldots, P$ and $j = 1, \ldots, M$,

$$w_{k,j}(x) = \Pr(W_k \leq x, S_k^* = j), \qquad r_{k,j}(x) = \Pr(R_k \leq x, S_k^* = j),$$

where W_k (resp. R_k) denotes a generic random variable representing the actual waiting time (resp. sojourn time) of class k customers in the steady state and S_k^* denotes a generic random variable representing the state of the underlying Markov chain immediately after arrivals of class k customers in the steady state. It then follows from the conditional PASTA (König and Schmidt 1990) that for $k = 1, \ldots, P$,

$$\boldsymbol{w}_k(x) = \frac{\boldsymbol{v}(x)\boldsymbol{D}_k}{\lambda_k}, \qquad \boldsymbol{r}_k(x) = \frac{\int_{0-}^{x} d\boldsymbol{v}(y)\boldsymbol{D}_k(x-y)}{\lambda_k}.$$

We denote by the LST's of $\boldsymbol{w}_k(x)$ and $\boldsymbol{r}_k(x)$ by $\boldsymbol{w}_k^*(s)$ and $\boldsymbol{r}_k^*(s)$, respectively. We then have

$$\boldsymbol{w}_k^*(s) = \frac{\boldsymbol{v}^*(s)\boldsymbol{D}_k}{\lambda_k}, \qquad \boldsymbol{r}_k^*(s) = \frac{\boldsymbol{v}^*(s)\boldsymbol{D}_k^*(s)}{\lambda_k}, \qquad Re(s) > 0, \tag{8}$$

for $k = 1, \ldots, P$.

Remark 1. $\boldsymbol{\kappa}$ represents the stationary probability vector of the underlying Markov chain given that the server is idle (Takine and Hasegawa 1994a), i.e.,

$$\Pr(V = 0, S = j) = (1 - \rho)\kappa_j, \qquad j = 1, \ldots, M,$$

where κ_j denotes the jth element of $\boldsymbol{\kappa}$. □

1 STATIONARY JOINT QUEUE LENGTH

This section consists of two subsections. Since customers are served on a FIFO basis, the number of *waiting* customers at time t is equal to the number of customers arriving in the attained waiting time at time t (Sengupta 1989, Sakasegawa and Wolff 1990). We thus consider the stationary attained waiting time in section 1.1. Next, in section 1.2. we derive the joint probability generating function of the stationary queue length distribution, by considering the number of customer arriving during the stationary attained waiting time.

1.1 Attained waiting time distribution

We start with the definition of the attained waiting time (Sengupta 1989). Let c_n denote the nth customer arriving to the system at time t_n. We denote the service time of c_n by H_n. Let V_t denote the amount of work in system at time t and $V_n = V_{t_n-}$ (i.e., the amount of work in system found by c_n on arrival). We denote by I_n the interval $[t_n + V_n, t_n + V_n + H_n)$ during which c_n is served. Then the attained waiting time at time t is defined as

$$A_t = t - t_n \quad \text{if} \quad t \in I_n,$$

and $A_t = 0$ if the system is empty. In other words, the attained waiting time is the length of time spent by a customer being served (if any) in the system. Thus all customers waiting in the queue arrived during the attained waiting time since customers are served on a FIFO basis. Let S_t be the state of the underlying Markov chain at time t and $S(n)$ be S_{t_n}. Assuming the steady state, we define $1 \times M$ vector $\boldsymbol{a}_k(x)$ ($k = 1, \ldots, P$) whose jth element $a_{k,j}(x)$ is given by

$$a_{k,j}(x) = \Pr(A_0 \le x, 0 \in I_n \text{ for some } n, S(n) = j, \mid c_n \text{ is of class } k).$$

Further we define $1 \times M$ vector $\boldsymbol{a}(x)$ whose jth element $a_j(x)$ is given by

$$a_j(x) = \Pr(A_0 = 0, S_0 = j) + \Pr(A_0 \le x, 0 \in I_n, S(n) = j).$$

Lemma 1. *$\boldsymbol{a}_k(x)$ and $\boldsymbol{a}(x)$ are given by*

$$\boldsymbol{a}_k(x) = \frac{\lambda_k}{\rho_k}\left[\int_{0-}^{x} \boldsymbol{w}_k(y)dy - \int_{0-}^{x} \boldsymbol{r}_k(y)dy\right]. \tag{9}$$

and

$$\boldsymbol{a}(x) = \boldsymbol{v}(x) + \int_{0-}^{x} \boldsymbol{v}(y)dy(\boldsymbol{C} + \boldsymbol{D}), \tag{10}$$

respectively.

Proof. Let $\boldsymbol{a}_k^*(s)$ be the LST of $\boldsymbol{a}_k(x)$:

$$\boldsymbol{a}_k^*(s) = \int_{0-}^{\infty} e^{-sx} d\boldsymbol{a}_k(x), \qquad Re(s) > 0,$$

whose jth element is denoted by $a_{k,j}^*(s)$. We call class k customers who arrive with the transition from state i to state j ($i.j = 1, \ldots, M$) class (k, i, j) customers. Note that $A > 0$ implies a customer is being served and the stationary probability of a class (k, i, j) customer being served is given by

$\rho_{k,i,j} = \pi_i D_{k,i,j} h_{k,i,j}$, where π_i denotes the ith element of $\boldsymbol{\pi}$. By definition, the waiting time of a customer being served and the elapsed service time are conditionally independent given a class (k,i,j) customer being served. Thus we have

$$a^*_{k,j}(s) = \sum_{i=1}^{M} w^*_{k.i.j}(s) \frac{1 - H^*_{k,i,j}(s)}{h_{k,i,j} s},$$

where $w^*_{k.i.j}(s)$ denotes the LST of the stationary waiting time distribution of class (k,i,j) customers and $H^*_{k,i,j}(s)$ denote the LST of the service time distribution function $H_{k,i,j}(x)$ of class (k,i,j) customers. Note here that with the conditional PASTA (König and Schmidt 1990),

$$w^*_{k,i,j}(s) = \frac{v^*_i(s) D_{k,i,j}}{\pi_i D_{k,i,j}}.$$

Thus we obtain

$$\begin{aligned} \boldsymbol{a}^*_k(s) &= \frac{\boldsymbol{v}^*(s)\boldsymbol{D}_k - \boldsymbol{v}^*(s)\boldsymbol{D}^*_k(s)}{\rho_k s} \\ &= \frac{\lambda_k(\boldsymbol{w}^*_k(s) - \boldsymbol{r}^*_k(s))}{\rho_k s}, \end{aligned} \tag{11}$$

from which (9) follows.

Let $\boldsymbol{a}^*(s)$ be the LST of $\boldsymbol{a}(x)$:

$$\boldsymbol{a}^*(s) = \int_{0-}^{\infty} e^{-sx} d\boldsymbol{a}(x), \qquad Re(s) > 0,$$

whose jth element is denoted by $a^*_j(s)$. It is clear from the definition of $\boldsymbol{a}(x)$ and Remark 1 that

$$\boldsymbol{a}(0) = (1 - \rho)\boldsymbol{\kappa}.$$

Thus

$$\boldsymbol{a}^*(s) = (1 - \rho)\boldsymbol{\kappa} + \sum_{k=1}^{P} \rho_k \boldsymbol{a}_k(x).$$

With (11), we obtain

$$\boldsymbol{a}^*(s) = (1 - \rho)\boldsymbol{\kappa} + \frac{\boldsymbol{v}^*(s)\boldsymbol{D} - \boldsymbol{v}^*(s)\boldsymbol{D}^*(s)}{s}. \tag{12}$$

Since $\boldsymbol{v}^*(s)$ satisfies (see (7))

$$\boldsymbol{v}^*(s)[s\boldsymbol{I} + \boldsymbol{C} + \boldsymbol{D}^*(s)] = (1 - \rho)\boldsymbol{\kappa} s, \tag{13}$$

(12) is rewritten to be

$$\boldsymbol{a}^*(s) = \boldsymbol{v}^*(s) + \frac{\boldsymbol{v}^*(s)}{s}(\boldsymbol{C} + \boldsymbol{D}),$$

from which (10) follows. □

Remark 2. Sengupta (1989) shows that the distribution of the attained waiting time in the stationary FIFO $G/G/1$ queue is equivalent to the stationary distribution of the virtual waiting time. In our formulation, this relationship is represented by $\boldsymbol{a}(x)\boldsymbol{e} = \boldsymbol{v}(x)\boldsymbol{e}$, which is readily confirmed with (1) and (10).

1.2 Joint queue length formulas

Let $\boldsymbol{x}^*(z_1, \ldots, z_P)$ denote a $1 \times M$ vector whose jth element $x_j(z_1, \ldots, z_P)$ is given by

$$x_j(z_1, \ldots, z_P) = E\left[z_1^{X_1} \cdots z_P^{X_P} 1_{\{S=j\}}\right], \qquad |z_k| \leq 1 \quad (k = 1, \ldots, P),$$

where X_k ($k = 1, \ldots, P$) denotes a generic random variable representing the number of waiting customers of class k in the steady state. Similarly, let $\boldsymbol{y}^*(z_1, \ldots, z_P)$ ($|z_k| \leq 1$ for all $k = 1, \ldots, P$) denote a $1 \times M$ vector whose jth element $y_j(z_1, \ldots, z_P)$ is given by

$$y_j(z_1, \ldots, z_P) = E\left[z_1^{Y_1} \cdots z_P^{Y_P} 1_{\{S=j\}}\right], \qquad |z_k| \leq 1 \quad (k = 1, \ldots, P),$$

where Y_k ($k = 1, \ldots, P$) denotes a generic random variable representing the number of class k customers in the steady state. To obtain $\boldsymbol{x}^*(z_1, \ldots, z_P)$ and $\boldsymbol{y}^*(z_1, \ldots, z_P)$, we need the following lemma.

Lemma 2. *The following equation holds for $|z_k| \leq 1$ ($k = 1, \ldots, P$).*

$$\begin{aligned}
&\sum_{k=1}^{P} \lambda_k z_k \int_{0-}^{\infty} d\boldsymbol{w}_k(x) \exp\left[\left(\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i\right) x\right] \\
&\quad = (1 - \rho)\boldsymbol{\kappa}\left(\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i\right) + \sum_{k=1}^{P} \lambda_k \int_{0-}^{\infty} d\boldsymbol{r}_k(x) \exp\left[\left(\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i\right) x\right]. \qquad (14)
\end{aligned}$$

Proof. It follows from (8) and (13) that

$$\boldsymbol{v}(x) + \int_{0-}^{x} \boldsymbol{v}(y)dy\boldsymbol{C} + \sum_{k=1}^{P} \lambda_k \int_{0-}^{x} \boldsymbol{r}_k(x)dy = (1 - \rho)\boldsymbol{\kappa}.$$

With this equation, we obtain

$$\begin{aligned}
(1 - \rho)\boldsymbol{\kappa} = &\int_{0-}^{\infty} d\boldsymbol{v}(x) \exp\left[\left(\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i\right) x\right] + \int_{0-}^{\infty} \boldsymbol{v}(x)\boldsymbol{C} \exp\left[\left(\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i\right) x\right] dx \\
&+ \sum_{k=1}^{P} \lambda_k \int_{0-}^{\infty} \boldsymbol{r}_k(x) \exp\left[\left(\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i\right) x\right] dx.
\end{aligned}$$

Postmultiplying both sides of the above equation by $\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i$ and rearranging terms, we obtain

$$(1-\rho)\boldsymbol{\kappa}\left(\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i\right) \tag{15}$$

$$\begin{aligned} &= \int_{0-}^{\infty} d\boldsymbol{v}(x) \exp\left[\left(\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i\right) x\right] \left(\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i\right) \\ &\quad + \int_{0-}^{\infty} \boldsymbol{v}(x)(\boldsymbol{C} + \boldsymbol{D}) \exp\left[\left(\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i\right) x\right] dx \left(\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i\right) \\ &\quad + \sum_{k=1}^{P} \lambda_k \int_{0-}^{\infty} (\boldsymbol{r}_k(x) - \boldsymbol{w}_k(x)) \exp\left[\left(\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i\right) x\right] dx \left(\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i\right), \end{aligned} \tag{16}$$

where we use $\boldsymbol{v}(x)\boldsymbol{D}_k = \lambda_k \boldsymbol{w}_k(x)$ and $\boldsymbol{D} = \sum_{k=1}^{P} \boldsymbol{D}_k$. Note here that

$$\lim_{x\to\infty} \boldsymbol{v}(x)(\boldsymbol{C} + \boldsymbol{D}) = \boldsymbol{\pi}(\boldsymbol{C} + \boldsymbol{D}) = \boldsymbol{0},$$

and

$$\lim_{x\to\infty} \boldsymbol{r}_k(x) = \lim_{x\to\infty} \boldsymbol{w}_k(x) = \boldsymbol{\pi}\boldsymbol{D}_k/\lambda_k.$$

Thus, taking partial integrals in the last two terms at the right hand side of (16) and noting $\exp[(\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i)x]$ and $\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i$ are commutative, we have

$$\begin{aligned} (1-\rho)\boldsymbol{\kappa}\left(\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i\right) &= \int_{0-}^{\infty} d\boldsymbol{v}(x) \left(\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i\right) \exp\left[\left(\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i\right) x\right] \\ &\quad - \int_{0-}^{\infty} d\boldsymbol{v}(x)(\boldsymbol{C} + \boldsymbol{D}) \exp\left[\left(\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i\right) x\right] \\ &\quad - \sum_{k=1}^{P} \lambda_k \int_{0-}^{\infty} d(\boldsymbol{r}_k(x) - \boldsymbol{w}_k(x)) \exp\left[\left(\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i\right) x\right] \\ &= \sum_{k=1}^{P} \lambda_k z_k \int_{0-}^{\infty} d\boldsymbol{w}_k(x) \exp\left[\left(\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i\right) x\right] \\ &\quad - \sum_{k=1}^{P} \lambda_k \int_{0-}^{\infty} d\boldsymbol{r}_k(x) \exp\left[\left(\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i\right) x\right], \end{aligned}$$

where we again use $\boldsymbol{v}(x)\boldsymbol{D}_k = \lambda_k \boldsymbol{w}_k(x)$ and $\boldsymbol{D} = \sum_{k=1}^{P} \boldsymbol{D}_k$ in the last equality. □

We now state the main theorem of the paper.

Theorem 1. *$\boldsymbol{x}^*(z_1, \ldots, z_P)$ and $\boldsymbol{y}^*(z_1, \ldots, z_P)$ satisfy*

$$\boldsymbol{x}^*(z_1, \ldots, z_P)\left(\boldsymbol{C} + \sum_{k=1}^{P} z_k \boldsymbol{D}_k\right) = \sum_{k=1}^{P} \lambda_k (z_k - 1) \int_{0-}^{\infty} d\boldsymbol{w}_k(x) \exp\left[\left(\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i\right) x\right],$$

and

$$\boldsymbol{y}^*(z_1, \ldots, z_P)\left(\boldsymbol{C} + \sum_{k=1}^{P} z_k \boldsymbol{D}_k\right) = \sum_{k=1}^{P} \lambda_k (z_k - 1) \int_{0-}^{\infty} d\boldsymbol{r}_k(x) \exp\left[\left(\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i\right) x\right],$$

respectively, for $|z_k| \leq 1$ *(*$k = 1, \ldots, P$*).*

Proof. By definition, $A_t = 0$ implies no customers in the system. Now suppose $A_t > 0$ and $t \in I_n$ for some n, i.e., c_n is being served. Since customers are served based on a FIFO basis, the number of *waiting* customers at time t is identical to the number of customers arriving during an interval $(t - A_t, t] = (t_n, t]$. Note that A_t is fully determined by arrivals before time t_n and arrivals during $(t_n, t]$ do not affect A_t. Further, given A_t and $S(n)$, the joint distribution of the number of customers of each class arriving in $(t_n, t]$ are completely determined. That is, if we define $\boldsymbol{x}^*(k, z_1, \ldots, z_P)$ and $\boldsymbol{y}^*(k, z_1, \ldots, z_P)$ as $1 \times M$ vectors whose jth elements $x_j(k, z_1, \ldots, z_P)$ and $y_j(k, z_1, \ldots, z_P)$ are given by

$$x_j(k, z_1, \ldots, z_P) = E\left[z_1^{X_1} \cdots z_P^{X_P} 1_{\{S=j\}} \mid \text{class } k \text{ customer is being served}\right],$$

and

$$y_j(k, z_1, \ldots, z_P) = E\left[z_1^{Y_1} \cdots z_P^{Y_P} 1_{\{S=j\}} \mid \text{a class } k \text{ customer is being served}\right],$$

respectively, we have

$$\boldsymbol{x}^*(k, z_1, \ldots, z_P) = \int_{0-}^{\infty} d\boldsymbol{a}_k(x) \exp\left[\left(\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i\right) x\right],$$
$$\boldsymbol{y}^*(k, z_1, \ldots, z_P) = z_k \int_{0-}^{\infty} d\boldsymbol{a}_k(x) \exp\left[\left(\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i\right) x\right].$$

Further Pr(a class k customer is being served) $= \rho_k$. These observations lead to

$$\boldsymbol{x}^*(z_1, \ldots, z_P) = (1 - \rho)\boldsymbol{\kappa} + \sum_{k=1}^{P} \rho_k \int_{0-}^{\infty} d\boldsymbol{a}_k(x) \exp\left[(\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i)x\right],$$

and

$$\boldsymbol{y}^*(z_1, \ldots, z_P) = (1 - \rho)\boldsymbol{\kappa} + \sum_{k=1}^{P} \rho_k z_k \int_{0-}^{\infty} d\boldsymbol{a}_k(x) \exp\left[(\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i)x\right].$$

With lemma 1, the above two equations are rewritten to be

$$\boldsymbol{x}^*(z_1, \ldots, z_P) = (1 - \rho)\boldsymbol{\kappa} + \sum_{k=1}^{P} \lambda_k \int_{0-}^{\infty} (\boldsymbol{w}_k(x) - \boldsymbol{r}_k(x)) \exp\left[(\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i)x\right] dx,$$

and

$$\boldsymbol{y}^*(z_1, \ldots, z_P) = (1 - \rho)\boldsymbol{\kappa} + \sum_{k=1}^{P} \lambda_k z_k \int_{0-}^{\infty} (\boldsymbol{w}_k(x) - \boldsymbol{r}_k(x)) \exp\left[(\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i)x\right] dx,$$

respectively. Postmultiplying both sides in these equations by $(\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i)$ and taking partial integrals, we obtain

$$\begin{aligned}
&\boldsymbol{x}^*(z_1, \ldots, z_P) \left(\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i \right) \\
&\qquad = (1 - \rho)\boldsymbol{\kappa} \left(\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i \right) - \sum_{k=1}^{P} \lambda_k \int_{0-}^{\infty} d(\boldsymbol{w}_k(x) - \boldsymbol{r}_k(x)) \exp \left[(\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i) x \right], \\
&\boldsymbol{y}^*(z_1, \ldots, z_P) \left(\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i \right) \\
&\qquad = (1 - \rho)\boldsymbol{\kappa} \left(\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i \right) - \sum_{k=1}^{P} \lambda_k z_k \int_{0-}^{\infty} d(\boldsymbol{w}_k(x) - \boldsymbol{r}_k(x)) \exp \left[(\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i) x \right].
\end{aligned}$$

Finally, with Lemma 2, we have the theorem. □

We now consider the total number of customers in the system and the marginal distribution of the number of class k customers in the system. Let $\overline{\boldsymbol{x}}^*(z)$ denote a $1 \times M$ vector whose jth element $\overline{x}_j(z)$ is given by

$$\overline{x}_j(z) = E \left[z^{X_1 + \cdots + X_P} 1_{\{S=j\}} \right], \qquad |z| \le 1.$$

Similarly, let $\overline{\boldsymbol{y}}^*(z)$ denote a $1 \times M$ vector whose jth element $\overline{y}_j(z)$ is given by

$$\overline{y}_j(z) = E \left[z^{Y_1 + \cdots + Y_P} 1_{\{S=j\}} \right], \qquad |z| \le 1.$$

By setting $z_k = z$ for all $k = 1, \ldots, P$ in Theorem 1, we obtain the following corollary.

Corollary 1. *$\overline{\boldsymbol{x}}^*(z)$ and $\overline{\boldsymbol{y}}^*(z)$ are given by*

$$\overline{\boldsymbol{x}}^*(z) (\boldsymbol{C} + z\boldsymbol{D}) = \lambda(z - 1) \sum_{k=1}^{P} \frac{\lambda_k}{\lambda} \int_{0-}^{\infty} d\boldsymbol{w}_k(x) \exp \left[(\boldsymbol{C} + z\boldsymbol{D})\, x \right],$$

and

$$\overline{\boldsymbol{y}}^*(z) (\boldsymbol{C} + z\boldsymbol{D}) = \lambda(z - 1) \sum_{k=1}^{P} \frac{\lambda_k}{\lambda} \int_{0-}^{\infty} d\boldsymbol{r}_k(x) \exp \left[(\boldsymbol{C} + z\boldsymbol{D})\, x \right],$$

respectively, where $|z| \le 1$.

Remark 3. The general relationship between the time-average queue length distribution and the queue length distribution immediately after departures in the stationary queue with (single-class) batch MAP arrivals has been established by Takine and Takahashi (1998). Note that Corollary 1 is considered as a special case of the result in Takine and Takahashi (1998), since the summands at the right hand sides of the equations in Corollary 1 represent the vector probability generating function of the

number of customers (regardless of their classes) immediately after departures of any class from the waiting space and from the system, respectively.

Also, let $\boldsymbol{x}_k^*(z)$ denote a $1 \times M$ vector whose jth element $x_{k,j}(z)$ is given by

$$x_{k,j}(z) = E\left[z^{X_k} 1_{\{S=j\}}\right], \qquad |z| \leq 1.$$

and let $\boldsymbol{y}_k^*(z)$ denote a $1 \times M$ vector whose jth element $y_{k,j}(z)$ is given by

$$y_{k,j}(z) = E\left[z^{Y_k} 1_{\{S=j\}}\right], \qquad |z| \leq 1.$$

By setting $z_k = z$ and $z_i = 1$ for $i = 1, \ldots, P, i \neq k$ in Theorem 1, we obtain the following corollary.

Corollary 2. *$\boldsymbol{x}_k^*(z)$ and $\boldsymbol{y}_k^*(z)$ ($k = 1, \ldots, P$) are given by*

$$\boldsymbol{x}_k^*(z)\,(\boldsymbol{C}_k + z\boldsymbol{D}_k) = \lambda_k(z-1)\int_{0-}^{\infty} d\boldsymbol{w}_k(x) \exp\left[(\boldsymbol{C}_k + z\boldsymbol{D}_k)\,x\right],$$

and

$$\boldsymbol{y}_k^*(z)\,(\boldsymbol{C}_k + z\boldsymbol{D}_k) = \lambda_k(z-1)\int_{0-}^{\infty} d\boldsymbol{r}_k(x) \exp\left[(\boldsymbol{C}_k + z\boldsymbol{D}_k)\,x\right],$$

respectively, where $|z| \leq 1$ and

$$\boldsymbol{C}_k = \boldsymbol{C} + \sum_{\substack{i=1 \\ i \neq k}}^{P} \boldsymbol{D}_i.$$

Remark 4. Corollary 2 is also considered as a special case of the result in Takine and Takahashi (1998), since the integrals at the right hand sides of the equations in Corollary 2 represent the vector probability generating function of the number of class k customers immediately after departures of class k from the waiting space and from the system, respectively.

2 RECURSION FORMULAS TO COMPUTE QUEUE LENGTH

In this section we provide recursion formulas to compute the queue length distributions. We first consider the joint distribution of the numbers of waiting customers in each class. Let $\boldsymbol{x}(n_1, \ldots, n_P)$ denote a $1 \times M$ vector whose jth element $x_j(n_1, \ldots, n_P)$ represents

$$x_j(n_1, \ldots, n_P) = \Pr(X_1 = n_1, \ldots, X_P = n_P, S = j).$$

By definition,

$$\boldsymbol{x}^*(z_1, \ldots, z_P) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_P=0}^{\infty} z_1^{n_1} \cdots z_P^{n_P}\, \boldsymbol{x}(n_1, \ldots, n_P).$$

Let θ denote the maximum absolute value of the diagonal elements of matrix $\boldsymbol{C}$. It then follows from Theorem 1 that

$$\sum_{n_1=0}^{\infty}\cdots\sum_{n_P=0}^{\infty} z_1^{n_1}\cdots z_P^{n_P}\boldsymbol{x}(n_1,\ldots,n_P)\left(\boldsymbol{C}+\sum_{k=1}^{P} z_k\boldsymbol{D}_k\right) = \sum_{k=1}^{P}(z_k-1)\sum_{m=0}^{\infty}\boldsymbol{v}^{(m)}(\theta)\boldsymbol{D}_k\left[\boldsymbol{I}+\theta^{-1}\left(\boldsymbol{C}+\sum_{i=1}^{P} z_i\boldsymbol{D}_i\right)\right]^m, \tag{17}$$

where

$$\boldsymbol{v}^{(m)}(\theta)=\int_{0-}^{\infty} e^{-\theta x}\frac{(\theta x)^m}{m!}d\boldsymbol{v}(x).$$

First we consider the computation of the $\boldsymbol{v}^{(m)}(\theta)$. Let $\boldsymbol{A}_m$ $(m = 0, 1, \ldots)$ denote

$$\boldsymbol{A}_0=\boldsymbol{I}+\theta^{-1}\left(\boldsymbol{C}+\boldsymbol{D}^{(0)}(\theta)\right), \qquad \boldsymbol{A}_m=\theta^{-1}\boldsymbol{D}^{(m)}(\theta), \qquad m=1,2,\ldots. \tag{18}$$

Lemma 3. *Let $\boldsymbol{G}$ denote an $M\times M$ stochastic matrix which is given by the minimum nonnegative solution of the equation:*

$$\boldsymbol{G}=\sum_{m=0}^{\infty}\boldsymbol{A}_m\boldsymbol{G}^m.$$

Let $\boldsymbol{K}$ denote an $M\times M$ stochastic matrix which is given by

$$\boldsymbol{K}=\boldsymbol{A}_0+\sum_{m=0}^{\infty}\boldsymbol{A}_m\boldsymbol{G}^{m-1},$$

and the invariant probability vector of $\boldsymbol{K}$ is denoted by $\boldsymbol{f}$. Then $\boldsymbol{v}^{(0)}(\theta)$ is given by

$$\boldsymbol{v}^{(0)}(\theta)=\frac{(1-\rho)}{\boldsymbol{f}\boldsymbol{A}_0\boldsymbol{e}}\boldsymbol{f},$$

and $\boldsymbol{v}^{(m)}(\theta)$'s are recursively computed by

$$\boldsymbol{v}^{(m)}(\theta)=\left(\boldsymbol{v}^{(0)}(\theta)\overline{\boldsymbol{B}}_m+\sum_{n=1}^{m-1}\boldsymbol{v}^{(n)}(\theta)\overline{\boldsymbol{A}}_{m-n+1}\right)\left(\boldsymbol{I}-\overline{\boldsymbol{A}}_1\right)^{-1}, \qquad m=1,2,\ldots,$$

where for $m = 1, 2, \ldots$,

$$\overline{\boldsymbol{A}}_m=\sum_{n=m}^{\infty}\boldsymbol{A}_n\boldsymbol{G}^{n-m}, \qquad \overline{\boldsymbol{B}}_m=\sum_{n=m}^{\infty}\boldsymbol{A}_{n+1}\boldsymbol{G}^{n-m}.$$

Proof. We note that $\boldsymbol{v}^{(m)}(\theta) > \boldsymbol{0}$,

$$\sum_{m=0}^{\infty}\boldsymbol{v}^{(m)}(\theta)z^m=\boldsymbol{v}^*(\theta-\theta z), \qquad |z|\le 1,$$

and therefore $\sum_{m=0}^{\infty} \boldsymbol{v}^{(m)}(\theta) = \boldsymbol{\pi}$. Roughly speaking, $\boldsymbol{v}^{(m)}(\theta)$'s represent the vector probability mass functions of m arrivals from an independent Poisson stream with rate θ during the stationary virtual waiting time whose vector LST is given by $\boldsymbol{v}^*(s)$.

From (13), we obtain

$$\boldsymbol{v}^*(\theta - \theta z)[\theta(1-z)\boldsymbol{I} + \boldsymbol{C} + \boldsymbol{D}^*(\theta - \theta z)] = \theta(1-z)(1-\rho)\boldsymbol{\kappa}, \tag{19}$$

Comparing coefficient vectors of z^m ($m = 0, 1, \ldots$) in both sides of (19), we have

$$\boldsymbol{v}^{(0)}(\theta)\left[\theta\boldsymbol{I} + \boldsymbol{C} + \boldsymbol{D}^{(0)}(\theta)\right] = \theta(1-\rho)\boldsymbol{\kappa}, \tag{20}$$

$$\boldsymbol{v}^{(1)}(\theta)\left[\theta\boldsymbol{I} + \boldsymbol{C} + \boldsymbol{D}^{(0)}(\theta)\right] + \boldsymbol{v}^{(0)}(\theta)\left[-\theta\boldsymbol{I} + \boldsymbol{D}^{(1)}(\theta)\right] = -\theta(1-\rho)\boldsymbol{\kappa}, \tag{21}$$

and for $m = 2, 3, \ldots,$

$$\boldsymbol{v}^{(m)}(\theta)\left[\theta\boldsymbol{I} + \boldsymbol{C} + \boldsymbol{D}^{(0)}(\theta)\right] + \boldsymbol{v}^{(m-1)}(\theta)\left[-\theta\boldsymbol{I} + \boldsymbol{D}^{(1)}(\theta)\right] + \sum_{n=0}^{m-2} \boldsymbol{v}^{(n)}(\theta)\boldsymbol{D}^{(m-n)}(\theta) = \boldsymbol{0}, \tag{22}$$

where

$$\boldsymbol{D}^{(m)}(\theta) = \int_{0-}^{\infty} e^{-\theta x}\frac{(\theta x)^m}{m!} d\boldsymbol{D}(x), \qquad m = 0, 1, \ldots.$$

From (20) and (21), we have

$$\boldsymbol{v}^{(0)}(\theta) = \boldsymbol{v}^{(1)}(\theta)\left[\boldsymbol{I} + \theta^{-1}\left(\boldsymbol{C} + \boldsymbol{D}^{(0)}(\theta)\right)\right] + \boldsymbol{v}^{(0)}(\theta)\left[\boldsymbol{I} + \theta^{-1}\left(\boldsymbol{C} + \boldsymbol{D}^{(0)}(\theta) + \boldsymbol{D}^{(1)}(\theta)\right)\right],$$

and from (22),

$$\boldsymbol{v}^{(m)}(\theta) = \boldsymbol{v}^{(m+1)}(\theta)\left[\boldsymbol{I} + \theta^{-1}\left(\boldsymbol{C} + \boldsymbol{D}^{(0)}(\theta)\right)\right] + \theta^{-1}\sum_{n=0}^{m} \boldsymbol{v}^{(n)}(\theta)\boldsymbol{D}^{(m-n)}(\theta), \qquad m = 1, 2, \ldots$$

That is, the above equations can be rewritten to be

$$\left(\boldsymbol{v}^{(0)}(\theta), \boldsymbol{v}^{(1)}(\theta), \ldots\right) = \left(\boldsymbol{v}^{(0)}(\theta), \boldsymbol{v}^{(1)}(\theta), \ldots\right) \begin{pmatrix} \boldsymbol{A}_0 + \boldsymbol{A}_1 & \boldsymbol{A}_2 & \boldsymbol{A}_3 & \boldsymbol{A}_4 & \cdots \\ \boldsymbol{A}_0 & \boldsymbol{A}_1 & \boldsymbol{A}_2 & \boldsymbol{A}_3 & \cdots \\ \boldsymbol{O} & \boldsymbol{A}_0 & \boldsymbol{A}_1 & \boldsymbol{A}_2 & \cdots \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A}_0 & \boldsymbol{A}_1 & \cdots \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{A}_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{23}$$

where $\boldsymbol{A}_m$'s ($m = 0, 1, \ldots$) are given in (18). Note here that $\boldsymbol{A}_m$'s ($m = 0, 1, \ldots$) are nonnegative $M \times M$ matrices which satisfies $\sum_{m=0}^{\infty} \boldsymbol{A}_m\boldsymbol{e} = \boldsymbol{e}$. Those and (23) imply that the $\boldsymbol{v}^{(m)}(\theta)$ is identical

with the stationary distribution of a Markov chain of $M/G/1$ type (Neuts 1989). Thus straightforward applications of the general theory of Markov chains of $M/G/1$ type (Neuts 1989) and a numerically stable recursion due to Ramaswami (1988) lead to Lemma 3. □

Remark 5. It can be shown that $\boldsymbol{G}$ introduced in Lemma 3 is given by

$$\boldsymbol{G} = \boldsymbol{I} + \theta^{-1}\boldsymbol{Q},$$

where $\boldsymbol{Q}$ is given in (5). Also, it follows from (20) that $\boldsymbol{v}^{(0)}(\theta)$ may be computed as

$$\boldsymbol{v}^{(0)}(\theta) = (1-\rho)\boldsymbol{\kappa}\left[\boldsymbol{I} + \theta^{-1}\left(\boldsymbol{C} + \boldsymbol{D}^{(0)}(\theta)\right)\right]^{-1},$$

when $\boldsymbol{I} + \theta^{-1}\left(\boldsymbol{C} + \boldsymbol{D}^{(0)}(\theta)\right)$ is nonsingular.

Next we consider the factor

$$\left[\boldsymbol{I} + \theta^{-1}\left(\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i\right)\right]^m.$$

We define $\boldsymbol{F}_m(n_1, \ldots, n_P)$'s for $m = 0, 1, \ldots$, $n_k \geq 0$ ($k = 1, \ldots, P$) and $n_1 + \cdots + n_P \leq m$ as $M \times M$ matrices which satisfy

$$\left[\boldsymbol{I} + \theta^{-1}\left(\boldsymbol{C} + \sum_{i=1}^{P} z_i \boldsymbol{D}_i\right)\right]^m = \sum_{\substack{n_1 \geq 0 \\ n_1 + \cdots + n_P \leq m}} \cdots \sum_{n_P \geq 0} z_1^{n_1} \cdots z_P^{n_P} \boldsymbol{F}_m(n_1, \ldots, n_P). \tag{24}$$

Lemma 4. *The $\boldsymbol{F}_m(n_1, \ldots, n_P)$ for $m = 0, 1, \ldots$, $n_k \geq 0$ ($k = 1, \ldots, P$) and $n_1 + \cdots + n_P \leq m$ is computed by the following recursion with $\boldsymbol{F}_0(0, \ldots, 0) = \boldsymbol{I}$:*
For $m = 0, 1, \ldots$,

$$\boldsymbol{F}_{m+1}(0, \ldots, 0) = (\boldsymbol{I} + \theta^{-1}\boldsymbol{C})^{m+1},$$

$$\boldsymbol{F}_{m+1}(n_1, \ldots, n_P) = \boldsymbol{F}_m(n_1, \ldots, n_P)(\boldsymbol{I} + \theta^{-1}\boldsymbol{C}), \\ + \sum_{\substack{k=1 \\ n_j \geq 1}}^{P} \boldsymbol{F}_m(n_1, \ldots, n_{k-1}, n_k - 1, n_{k+1}, \ldots, n_P)\theta^{-1}\boldsymbol{D}_k, \\ \text{if } \sum_{k=1}^{P} n_k = 1, 2, \ldots, m,$$

$$\boldsymbol{F}_{m+1}(0, \ldots, 0, \underbrace{m+1}_{k\text{th}}, 0, \ldots, 0) = (\theta^{-1}\boldsymbol{D}_k)^{m+1}.$$

Proof. Note that for $m = 0, 1, \ldots$,

$$\begin{aligned}
&\sum_{\substack{n_1\geq 0\\ n_1+\cdots+n_P\leq m+1}} \cdots \sum_{n_P\geq 0} z_1^{n_1}\cdots z_P^{n_P}\boldsymbol{F}_{m+1}(n_1,\ldots,n_P)\\
&\quad = \sum_{\substack{n_1\geq 0\\ n_1+\cdots+n_P\leq m}} \cdots \sum_{n_P\geq 0} z_1^{n_1}\cdots z_P^{n_P}\boldsymbol{F}_m(n_1,\ldots,n_P)\left[\boldsymbol{I}+\theta^{-1}\left(\boldsymbol{C}+\sum_{i=1}^{P} z_i\boldsymbol{D}_i\right)\right]\\
&\quad = \sum_{\substack{n_1\geq 0\\ n_1+\cdots+n_P\leq m}} \cdots \sum_{n_P\geq 0} z_1^{n_1}\cdots z_P^{n_P}\boldsymbol{F}_m(n_1,\ldots,n_P)\left[\boldsymbol{I}+\theta^{-1}\boldsymbol{C}\right]\\
&\qquad + \sum_{k=1}^{P}\sum_{\substack{n_1\geq 0\\ n_1+\cdots+n_P\leq m}} \cdots \sum_{n_P\geq 0} z_1^{n_1}\cdots z_{k-1}^{n_{k-1}} z_k^{n_k+1} z_{k+1}^{n_{k+1}}\cdots z_P^{n_P}\boldsymbol{F}_m(n_1,\ldots,n_P)\theta^{-1}\boldsymbol{D}_k. \qquad (25)
\end{aligned}$$

Comparing the coefficient vectors of $z_1^{n_1}\cdots z_P^{n_P}$ of both sides of the above equation, we obtain the lemma. □

Remark 6. Lemma 4 is considered as an extension of the results in Lucantoni and Ramaswami (1985) and Takine et al. (1994b) to a multi-dimensional case.

Let $\boldsymbol{D}_k^{(m)}(\theta)$ be

$$\boldsymbol{D}_k^{(m)}(\theta) = \int_{0-}^{\infty} e^{-\theta x}\frac{(\theta x)^m}{m!}d\boldsymbol{D}_k(x), \qquad k = 1,\ldots,P, m = 0,1,\ldots.$$

Together with the above results, we obtain the following theorem.

Theorem 2. *The vector joint distribution $\boldsymbol{x}(n_1,\ldots,n_P)$ for $n_k = 0, 1, \ldots$ ($k = 1,\ldots,P$) is given by*

$$\boldsymbol{x}(0,\ldots,0) = \sum_{m=0}^{\infty}\boldsymbol{v}^{(m)}(\theta)\boldsymbol{D}(\boldsymbol{I}+\theta\boldsymbol{C})^m(-\boldsymbol{C})^{-1},$$

and for $n_1+\cdots+n_P \geq 1$,

$$\begin{aligned}
&\boldsymbol{x}(n_1,\ldots,n_P)\\
&\quad = \Bigg[\sum_{\substack{k=1\\ n_k\geq 1}}^{P}\boldsymbol{x}(n_1,\ldots,n_{k-1},n_k-1,n_{k+1},\ldots,n_P)\boldsymbol{D}_k\\
&\qquad + \sum_{m=n_1+\cdots+n_P}^{\infty}\boldsymbol{v}^{(m)}(\theta)\boldsymbol{D}\boldsymbol{F}_m(n_1,\ldots,n_P)\\
&\qquad - \sum_{\substack{k=1\\ n_k\geq 1}}^{P}\sum_{m=n_1+\cdots+n_P-1}^{\infty}\boldsymbol{v}^{(m)}(\theta)\boldsymbol{D}_k\boldsymbol{F}_m(n_1,\ldots,n_{k-1},n_k-1,n_{k+1},\ldots,n_P)\Bigg](-\boldsymbol{C})^{-1}.
\end{aligned}$$

Also, the vector joint distribution $\boldsymbol{y}(n_1, \ldots, n_P)$ *for* $n_k = 0, 1, \ldots$ $(k = 1, \ldots, P)$ *is given by*

$$\boldsymbol{y}(0, \ldots, 0) = (1 - \rho)\boldsymbol{\kappa},$$

and for $n_1 + \cdots + n_P \geq 1$,

$$\begin{aligned}
&\boldsymbol{y}(n_1, \ldots, n_P) \\
&\quad = \Bigg[\sum_{\substack{k=1 \\ n_k \geq 1}}^{P} \boldsymbol{y}(n_1, \ldots, n_{k-1}, n_k - 1, n_{k+1}, \ldots, n_P) \boldsymbol{D}_k \\
&\qquad + \sum_{m=n_1+\cdots+n_P}^{\infty} \sum_{i=0}^{m} \boldsymbol{v}^{(i)}(\theta) \boldsymbol{D}^{(m-i)}(\theta) \boldsymbol{F}_m(n_1, \ldots, n_P) \\
&\qquad - \sum_{\substack{k=1 \\ n_k \geq 1}}^{P} \sum_{m=n_1+\cdots+n_P-1}^{\infty} \sum_{i=0}^{m} \boldsymbol{v}^{(i)}(\theta) \boldsymbol{D}_k^{(m-i)}(\theta) \boldsymbol{F}_m(n_1, \ldots, n_{k-1}, n_k - 1, n_{k+1}, \ldots, n_P) \Bigg] (-\boldsymbol{C})^{-1},
\end{aligned}$$

where the $\boldsymbol{v}^{(m)}(\theta)$ *is computed with Lemma 3, and the* $\boldsymbol{F}_m(n_1, \ldots, n_P)$ *is computed with Lemma 4.*

Proof. From (17) and (24), we have

$$\begin{aligned}
&\sum_{n_1=0}^{\infty} \cdots \sum_{n_P=0}^{\infty} z_1^{n_1} \cdots z_P^{n_P} \boldsymbol{x}(n_1, \ldots, n_P) \left(\boldsymbol{C} + \sum_{k=1}^{P} z_k \boldsymbol{D}_k \right) \\
&\qquad = \sum_{k=1}^{P} (z_k - 1) \sum_{m=0}^{\infty} \boldsymbol{v}^{(m)}(\theta) \boldsymbol{D}_k \sum_{\substack{n_1 \geq 0 \\ n_1+\cdots+n_P \leq m}} \cdots \sum_{n_P \geq 0} z_1^{n_1} \cdots z_P^{n_P} \boldsymbol{F}_m(n_1, \ldots, n_P) \\
&\qquad = \sum_{k=1}^{P} (z_k - 1) \sum_{n_1=0}^{\infty} \cdots \sum_{n_P=0}^{\infty} z_1^{n_1} \cdots z_P^{n_P} \sum_{m=n_1+\cdots+n_P}^{\infty} \boldsymbol{v}^{(m)}(\theta) \boldsymbol{D}_k \boldsymbol{F}_m(n_1, \ldots, n_P).
\end{aligned}$$

Comparing the coefficient vectors of $z_1^{n_1} \cdots z_P^{n_P}$ of both sides of the above equation, we obtain the former part of the theorem. The equation for $\boldsymbol{y}(0, \ldots, 0)$ follows from Remark 1. The recursion for the $\boldsymbol{y}(n_1, \ldots, n_P)$ can be proved in a very similar way by noting

$$\lambda_k \int_{0-}^{\infty} e^{-\theta x} \frac{(\theta x)^m}{m!} d\boldsymbol{r}_k(x) = \sum_{i=0}^{m} \boldsymbol{v}^{(i)}(\theta) \boldsymbol{D}_k^{(m-i)}(\theta).$$

Thus we obtain

$$\begin{aligned}
&\sum_{n_1=0}^{\infty} \cdots \sum_{n_P=0}^{\infty} z_1^{n_1} \cdots z_P^{n_P} \boldsymbol{y}(n_1, \ldots, n_P) \left(\boldsymbol{C} + \sum_{k=1}^{P} z_k \boldsymbol{D}_k \right) \\
&\qquad = \sum_{k=1}^{P} (z_k - 1) \sum_{n_1=0}^{\infty} \cdots \sum_{n_P=0}^{\infty} z_1^{n_1} \cdots z_P^{n_P} \sum_{m=n_1+\cdots+n_P}^{\infty} \sum_{i=0}^{m} \boldsymbol{v}^{(i)}(\theta) \boldsymbol{D}_k^{(m-i)}(\theta) \boldsymbol{F}_m(n_1, \ldots, n_P),
\end{aligned}$$

from which the recursion for the $\boldsymbol{y}(n_1, \ldots, n_P)$ follows. □

Next we consider the total number of waiting customers and the total number of customers in the system. Let $\overline{\boldsymbol{x}}(n)$ and $\overline{\boldsymbol{y}}(n)$ $(n = 0, 1, \ldots)$ denote $1 \times M$ vectors whose j elements $\overline{x}_j(n)$ and $\overline{y}_j(n)$ represent

$$\overline{x}_j(n) = \Pr(X = n, S = j), \qquad \overline{y}_j(n) = \Pr(Y = n, S = j),$$

respectively, where $X = X_1 + \cdots + X_P$ and $Y = Y_1 + \cdots + Y_P$.

Corollary 3. *The vector distribution $\boldsymbol{x}(n)$ is given by*

$$\overline{\boldsymbol{x}}(0) = \sum_{m=0}^{\infty} \boldsymbol{v}^{(m)}(\theta)\boldsymbol{D}(\boldsymbol{I} + \theta\boldsymbol{C})^m(-\boldsymbol{C})^{-1},$$

and for $n = 1, 2, \ldots$,

$$\overline{\boldsymbol{x}}(n) = \left[\overline{\boldsymbol{x}}(n-1)\boldsymbol{D} + \sum_{m=n}^{\infty} \boldsymbol{v}^{(m)}(\theta)\boldsymbol{D}\boldsymbol{F}_m(n) - \sum_{m=n-1}^{\infty} \boldsymbol{v}^{(m)}(\theta)\boldsymbol{D}\boldsymbol{F}_m(n-1)\right](-\boldsymbol{C})^{-1},$$

where the $\boldsymbol{v}^{(m)}(\theta)$ is computed with Lemma 3, and the $\boldsymbol{F}_m(n)$ is computed with $\boldsymbol{F}_0(0) = \boldsymbol{I}$ and for $m = 0, 1, \ldots$

$$\boldsymbol{F}_{m+1}(n) = \begin{cases} (\boldsymbol{I} + \theta^{-1}\boldsymbol{C})^{m+1}, & n = 0, \\ \boldsymbol{F}_m(n)(\boldsymbol{I} + \theta^{-1}\boldsymbol{C}) + \boldsymbol{F}_m(n-1)\theta^{-1}\boldsymbol{D}, & n = 1, \ldots, m, \\ (\theta^{-1}\boldsymbol{D})^{m+1}, & n = m+1. \end{cases}$$

Further the vector distribution $\overline{\boldsymbol{y}}(n)$ is given by

$$\begin{aligned} \overline{\boldsymbol{y}}(0) &= (1-\rho)\boldsymbol{\kappa}, \\ \overline{\boldsymbol{y}}(1) &= \overline{\boldsymbol{x}}(0) - \overline{\boldsymbol{y}}(0), \\ \overline{\boldsymbol{y}}(n) &= \overline{\boldsymbol{x}}(n-1), \qquad n = 2, 3, \ldots, \end{aligned}$$

Proof. Since the corollary for the $\overline{\boldsymbol{x}}(n)$ can be proved in the same way as Theorem 2, so that we omit the details. The results for the $\overline{\boldsymbol{y}}(n)$ is clear from the definitions of $\overline{\boldsymbol{x}}(n)$ and $\overline{\boldsymbol{y}}(n)$. □

Finally we consider the marginal distributions of the number of waiting customers of class k and the the number of class k customers in the system. Let $\boldsymbol{x}_k(n)$ and $\boldsymbol{y}_k(n)$ $(n = 0, 1, \ldots)$ denote $1 \times M$ vectors whose j elements $x_{k,j}(n)$ and $y_{k,j}(n)$ represent

$$x_{k,j}(n) = \Pr(X_k = n, S = j), \qquad y_{k,j}(n) = \Pr(Y_k = n, S = j),$$

respectively.

Corollary 4. *The vector distribution $\boldsymbol{x}_k(n)$ is given by*

$$\boldsymbol{x}_k(0) = \sum_{m=0}^{\infty} \boldsymbol{v}^{(m)}(\theta)\boldsymbol{D}_k(\boldsymbol{I} + \theta^{-1}\boldsymbol{C}_k)^m(-\boldsymbol{C}_k)^{-1},$$

and for $n = 1, 2, \ldots$,

$$\boldsymbol{x}_k(n) = \left[\boldsymbol{x}_k(n-1)\boldsymbol{D} + \sum_{m=n}^{\infty} \boldsymbol{v}^{(m)}(\theta)\boldsymbol{D}_k\boldsymbol{F}_m^{(k)}(n) - \sum_{m=n-1}^{\infty} \boldsymbol{v}^{(m)}(\theta)\boldsymbol{D}_k\boldsymbol{F}_m^{(k)}(n-1)\right](-\boldsymbol{C}_k)^{-1},$$

where the $\boldsymbol{v}^{(m)}(\theta)$ is computed with Lemma 3, and the $\boldsymbol{F}_m^{(k)}(n)$ is computed with $\boldsymbol{F}_0^{(k)}(0) = \boldsymbol{I}$ and for $m = 0, 1, \ldots$

$$\boldsymbol{F}_{m+1}^{(k)}(n) = \begin{cases} (\boldsymbol{I} + \theta^{-1}\boldsymbol{C}_k)^{m+1}, & n = 0, \\ \boldsymbol{F}_m^{(k)}(n)(\boldsymbol{I} + \theta^{-1}\boldsymbol{C}_k) + \boldsymbol{F}_m^{(k)}(n-1)\theta^{-1}\boldsymbol{D}_k, & n = 1, \ldots, m, \\ (\theta^{-1}\boldsymbol{D}_k)^{m+1}, & n = m+1. \end{cases}$$

Further the vector distribution $\boldsymbol{y}_k(n)$ for $n = 0, 1, \ldots$ is given by

$$\begin{aligned} \boldsymbol{y}_k(0) &= \sum_{m=0}^{\infty}\sum_{n=0}^{m} \boldsymbol{v}^{(n)}(\theta)\boldsymbol{D}_k^{(m-n)}(\theta)(\boldsymbol{I} + \theta^{-1}\boldsymbol{C}_k)^m\boldsymbol{D}_k(-\boldsymbol{C}_k)^{-1}, \\ \boldsymbol{y}_k(1) &= \boldsymbol{x}_k(0) - \boldsymbol{y}_k(0), \\ \boldsymbol{y}_k(n) &= \boldsymbol{x}_k(n-1), \qquad n = 2, 3, \ldots. \end{aligned}$$

Since Corollary 4 can be proved in the same way as Corollary 3, we omit the proof.

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