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Interior and exterior functions of positive Boolean functions

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Abstract

The interior function and exterior function of a Boolean function f are introduced [14] in order to express its stability and robustness properties. In this paper, we investigate the complexity of two problems α-INTERIOR and α-EXTERIOR. We first answer the question about the complexity of α-INTERIOR left open in [14]; it has no polynomial total time algorithm even if α is bounded by a constant, unless P=NP. However, for positive h-term DNF functions with h bounded by a constant, problem α-INTERIOR and α-EXTERIOR can be solved in (input) polynomial time and polynomial delay, respectively. Furthermore, for positive k-DNF functions, α-INTERIOR for two cases in which $k = 1$, and α and k are both bounded by a constant, can be solved in polynomial delay and in polynomial time, respectively.

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1 Definitions and Introduction

A *Boolean function*, or a *function* in short, is a mapping $f : \{0,1\}^n \mapsto \{0,1\}$, where $v \in \{0,1\}^n$ is called a *Boolean vector* (a *vector* in short). If $f(v) = 1$ (resp., 0), then v is called a *true* (resp., *false*) vector of f. The set of all true vectors (resp., false vectors) is denoted by $T(f)$ (resp., $F(f)$). Two special functions with $T(f) = \emptyset$ and $F(f) = \emptyset$ are respectively denoted by $f = \perp$ and $f = \top$. In this paper, we assume $f \neq \top, \perp$. Denote, for a vector $v \in \{0,1\}^n$, $ON(v) = \{j \mid v_j = 1,\ j = 1,2,\ldots,n\}$ and $OFF(v) = \{j \mid v_j = 0,\ j = 1,2,\ldots,n\}$. The *$\alpha$-neighborhood of* a vector v is defined by

$$N_\alpha(v) = \{w \mid \parallel w - v \parallel \leq \alpha\},$$

where $\parallel y \parallel$ denotes $\sum_{i=1}^{n} |y_i|$. For two functions f and g on the same set of variables, we write $f \leq g$ if $f(v) = 1$ implies $g(v) = 1$ for all $v \in \{0,1\}^n$, and $f < g$ if $f \leq g$ and $f \neq g$. For two vectors $v, w \in \{0,1\}^n$, we write $v \leq w$ if $v_i \leq w_i$ holds for all i, and write $v < w$ if $v \leq w$ and $v \neq w$. A function f is *positive* if $v \leq w$ always implies $f(v) \leq f(w)$. A positive function is also called *monotone*. A true vector v of f is *minimal* if there is no other true vector w such that $w < v$, and let $\min T(f)$ denote the set of all minimal true vectors of f. A *maximal* false vector is symmetrically defined and $\max F(f)$ denotes the set of all maximal false vectors of f.

For a Boolean function f of n variables and a nonnegative integer α, the *α-interior function* $\sigma_{-\alpha}(f)$ and the *α-exterior function* $\sigma_\alpha(f)$ are respectively defined by

$$\sigma_{-\alpha}(f)(v) = \begin{cases} 1, & \text{if } N_\alpha(v) \subseteq T(f) \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\sigma_\alpha(f)(v) = \begin{cases} 1, & \text{if } N_\alpha(v) \cap T(f) \neq \phi \\ 0, & \text{otherwise,} \end{cases}$$

for $v \in \{0,1\}^n$, where $\sigma_0(f) = f$ holds by definition. Furthermore, the *θ-layer* of f is defined by

$$L_\theta(f) \;=\; T(\sigma_\theta(f)) \setminus T(\sigma_{\theta-1}(f)),$$

where θ is an integer.

¿From these definitions, we have the following properties [14]:

(i) $\sigma_\theta(f) \leq \sigma_\eta(f)$ holds for integers $\theta \leq \eta$.

(ii) $\sigma_\theta(f) \leq \sigma_\theta(g)$ holds for functions $f \leq g$ and an integer θ.

(iii) $\sigma_\alpha(\sigma_\beta(f)) = \sigma_{\alpha+\beta}(f)$ and $\sigma_{-\alpha}(\sigma_{-\beta}(f)) = \sigma_{-\alpha-\beta}(f)$ hold for nonnegative integers α and β.

The *dual* of a function f, denoted f^d, is defined by

$$f^d(x) = \bar{f}(\bar{x}),$$

where $\bar{f}$ and $\bar{x}$ denote the complement of f and x, respectively. As is well-known, a Boolean expression of f^d is obtained from a Boolean expression of f by exchanging $\vee$ (or) and $\cdot$ (and), as well as the constants 0 and 1. It is easy to see that $(f \vee g)^d = f^d g^d$, $(fg)^d = f^d \vee g^d$, $f \leq g$ implies $f^d \geq g^d$, and so on. Concerning our functions, we have

(iv) $\sigma_\theta(f^d) = (\sigma_{-\theta}(f))^d$ holds for any integer θ.

As an example of usages of σ_θ, assume that $f(v)$ describes the result of nutrient-headache relationship for a patient [5, 8], e.g., $f(01101) = 1$ (resp., 0) denotes that the patient has (resp., does not have) a headache when he consumes foods $\sharp 2, \sharp 3$ and $\sharp 5$ but neither $\sharp 1$ nor $\sharp 4$. Now, take a vector $v \in L_\theta(f)$. If $\theta \leq 0$, it means that the patient still has a headache even if he changes any $-\theta$ foods in v, but will not have a headache if he appropriately changes $-\theta + 1$ foods. If $\theta > 0$, it means that he does not have a headache even if he changes $\theta - 1$ foods but may have a headache if he changes θ foods.

Another example is found in the network reliability problem [2, 6]. For a connected graph $G = (V, E)$ with $E = \{1, 2, \ldots, n\}$, introduce a vector $v \in \{0, 1\}^n$, where v_i corresponds to $i \in E$. We interpret that edge i can be in one of the two states: *operative* ($v_i = 1$) or *failed* ($v_i = 0$). Then let $G_v = (V, ON(v))$ and define the *spanning tree function* f of G by

$$f(v) = \begin{cases} 1, & \text{if } G_v \text{ is connected} \\ 0, & \text{otherwise.} \end{cases}$$

This f is a positive function, and $\min T(f)$ represents the set of all *spanning trees* of G. Now, let $v \in L_\theta(f)$. If $\theta \leq 0$, then it denotes that graph G_v is $(-\theta + 1)$-edge connected. If $\theta > 0$, then it denotes that graph G_v is not connected but becomes connected if appropriate θ edges are repaired.

In the network reliability problem, a number of characteristics, which have to be analyzed, can be represented by positive functions; e.g., [2, 6, 18]. Positive functions have also been studied in such fields as learning theory [1], game theory [18] and hypergraph theory [3].

In the rest of this section, we review fundamental properties of interior and exterior functions of positive functions, studied in [14]. It is known that a positive function f has the unique minimal disjunctive normal form (DNF), consisting of all prime implicants, which correspond one-to-one to minimal true vectors. For example, a positive function $f = x_1x_2 \vee x_2x_3 \vee x_3x_1$ has prime implicants x_1x_2, x_2x_3, x_3x_1, which correspond to minimal true vectors (110), (011), (101), respectively. In other words, the input length to describe a positive function f is $O(n|\min T(f)|)$ if it is represented in this manner. If f is a positive function, then $\sigma_\theta(f)$ is also positive for any integer θ. Let $e^{(k)}$ denote the k-th unit vector; i.e., $e_i^{(k)} = 1$ if $i = k$, otherwise $e_i^{(k)} = 0$. Then all interior and exterior functions can be given as follows.

$$\sigma_{-\alpha}(f)(v) = \begin{cases} 1, & \text{if } v - \sum_{j \in S} e^{(j)} \in T(f) \\ & \quad \text{for all } S \subseteq ON(v) \text{ such that } |S| = \min\{\alpha, |ON(v)|\} \\ 0, & \text{otherwise,} \end{cases} \tag{1.1}$$

$$\sigma_\alpha(f)(v) = \begin{cases} 1, & \text{if } v + \sum_{j\in S} e^{(j)} \in T(f) \\ & \quad \text{for some } S \subseteq OFF(v) \text{ such that } |S| = \min\{\alpha, |OFF(v)|\} \\ 0, & \text{otherwise.} \end{cases} \tag{1.2}$$

The next theorem characterizes $\max F(\sigma_{-\alpha}(f))$ and $\min T(\sigma_\alpha(f))$;

Theorem 1.1 ([14]) Let f be a positive function and α be a nonnegative integer. Then

(i) $\max F(\sigma_{-\alpha}(f)) = \text{MaxSet}\{b + \sum_{j\in S} e^{(j)} \mid b \in \max F(f), \; S \subseteq OFF(b), |S| = \min\{\alpha, |OFF(b)|\}\}$

(ii) $\min T(\sigma_\alpha(f)) = \text{MinSet}\{a - \sum_{j\in S} e^{(j)} \mid a \in \min T(f), \; S \subseteq ON(a), |S| = \min\{\alpha, |ON(a)|\}\}$,

where $\text{MinSet}(A)$ (resp., $\text{MaxSet}(A)$) denotes the set of minimal (resp., maximal) vectors in set A.

However, we note that this theorem is not easily extendable to derive similar forms of $\min T(\sigma_{-\alpha}(f))$ and $\max F(\sigma_\alpha(f))$.

2 Computing interior and exterior functions of positive functions

The following problems α-INTERIOR and α-EXTERIOR were introduced in [14] and studied mainly from the view-point of their computational complexity.

Problem α-INTERIOR (resp., α-EXTERIOR)
Input: $\min T(f)$, where f is a positive function of n variables, and a nonnegative integer α.
Output: $\min T(\sigma_{-\alpha}(f))$ (resp., $\min T(\sigma_\alpha(f))$).

In these problems, it is asked to enumerate all vectors in $\min T(\cdot)$. Different from decision problems, the complexity of enumeration problems is usually evaluated on their input and output length. An algorithm is called *polynomial total time* if its running time is polynomial in the length of input and output, and is called *polynomial delay* if the time between consecutive outputs is bounded by a polynomial in the input length, and the first (resp., last) output occurs also in polynomial time after (resp., before) the start (resp., halt) of the algorithm [12]. A polynomial delay algorithm is also polynomial total time, but the converse may not be true.

The following properties were shown in [14]:

(I) There is no polynomial total time algorithms for α-INTERIOR, unless P=NP.

(II) There is no polynomial total time algorithms for α-EXTERIOR, unless P=NP.

(III) For a nonnegative integer α bounded by a constant, α-EXTERIOR can be solved in input polynomial time.

(IV) If α-INTERIOR has a polynomial total time algorithm for a nonnegative integer α bounded by a constant, then there is a polynomial total time algorithm for dualizing a positive Boolean function.

Here, the problem of dualizing a positive Boolean function asks to output $\max F(f)$ from $\min T(f)$ of a given positive function f. This problem has been intensively studied [4, 9, 12, 13], and there is an $O(m^{o(\log m)})$ time algorithm, where $m = |\min T(f)| + |\max F(f)|$, found by M. L. Fredman and L. Khachiyan [10].

In this paper we first strengthen the result (IV), and show that, even if α is bounded by a constant, there is no polynomial total time algorithm for problem α-INTERIOR, unless P=NP.

We further study some special classes of positive functions. For example, it is known [14] that the class of 2-monotonic positive functions $\mathcal{C}_{2M}$ [7, 15, 17] has the following nice property.

(V) There are incrementally polynomial time algorithms for problems α-INTERIOR($\mathcal{C}_{2M}$) and α-EXTERIOR($\mathcal{C}_{2M}$).

Here, α-INTERIOR($\mathcal{C}$) (resp., α-EXTERIOR($\mathcal{C}$)) denotes the problem α-INTERIOR (resp., α-EXTERIOR) restricted to the functions in class $\mathcal{C}$. In this paper, we consider two other subclasses of positive functions $\mathcal{C}_{h\text{-}term}$ and $\mathcal{C}_{k\text{-}DNF}$, which respectively denote the classes of h-term DNF and k-DNF positive functions. These functions has been well studied; e.g., [19, 1, 3, 5]. We show that, if h is bounded by a constant, problems α-INTERIOR($\mathcal{C}_{h\text{-}term}$) and α-EXTERIOR($\mathcal{C}_{h\text{-}term}$) have (input) polynomial time and polynomial delay algorithms, respectively. As for positive k-DNF functions f, it is easy to see that $\sigma_\alpha(f) = \top$ holds for all $\alpha \geq k$. It follows from this and (III) that for any k bounded by a constant, problem α-EXTERIOR($\mathcal{C}_{k\text{-}DNF}$) can be solved in input polynomial time. Furthermore it was shown in [14] that, for general α and fixed $k \geq 2$, α-INTERIOR($\mathcal{C}_{k\text{-}DNF}$) has no polynomial total time algorithm, unless P=NP. In this paper, we present a polynomial delay algorithm for α-INTERIOR($\mathcal{C}_{1\text{-}DNF}$) and show that for any k bounded by a constant, α-INTERIOR($\mathcal{C}_{k\text{-}DNF}$) can be solved in (input) polynomial time, if α is also bounded by a constant.

Table 1 summarizes the results in this paper.

The rest of the paper is organized as follows. Section 3 studies problem α-INTERIOR with α bounded by a constant. In Sections 4 and 5, we consider h-term DNF positive functions and k-DNF positive functions respectively, and show that there exist polynomial total time algorithms for α-INTERIOR in some cases. Finally, Section 6 concludes this paper.

3 Problem α-INTERIOR for α bounded by a constant

Although we can obtain a vector $v \in \min T(\sigma_{-\alpha}(f))$ from $\min T(f)$ in polynomial time [14], it will be shown below that generating all vectors $v \in \min T(\sigma_{-\alpha}(f))$ (i.e., problem α-INTERIOR) is intractable, unless P=NP, even if α is bounded by a constant.

Table 1: Summary of results.

Function classes*	α-INTERIOR		α-EXTERIOR	
	α: general	α: constant	α: general	α: constant
Positive	NPT [14]	NPT	NPT [14]	P [14]
Positive 1-DNF	PD	P	P [14]	P [14]
Positive k-DNF with fixed $k \geq 2$	NPT [14]	P	P	P [14]
Positive h-term DNF with fixed $h \geq 2$	P	P	PD	P [14]

*In these classes, k and h are bounded by constants.

P: Polynomial time

PD: Polynomial delay

NPT: No polynomial total time algorithm unless P=NP

Problem ADDITIONAL α-INTERIOR
Input: $\min T(f)$, and a set $P \subseteq \min T(\sigma_{-\alpha}(f))$, where f is a positive function of n variables.
Question: $\min T(\sigma_{-\alpha}(f)) \setminus P \neq \emptyset$?

Lemma 3.1 Given an α (≥ 1) bounded by a constant, problem ADDITIONAL α-INTERIOR is NP-complete.

Proof. Let us first show that problem ADDITIONAL α-INTERIOR is in NP. Since $f \neq \top$ by assumption, any $v \in T(\sigma_{-\alpha}(f))$ satisfies $|ON(v)| > \alpha$. Thus we can conclude that $v \in \min T(\sigma_{-\alpha}(f))$ holds if and only if the following two conditions hold:

(c1) $v - \sum_{l \in S} e^{(l)} \in T(f)$ for all $S \subseteq ON(v)$ such that $|S| = \alpha$. (This checks if $v \in T(\sigma_{-\alpha}(f))$ by (1.1).)

(c2) For every $i \in ON(v)$, there exists an $S \subseteq ON(v - e^{(i)})$ such that $|S| = \alpha$ and $(v - e^{(i)}) - \sum_{l \in S} e^{(l)} \in F(f)$. (This checks if v is minimal in $T(\sigma_{-\alpha}(f))$ by (1.1).)

In addition, to check if $v \notin P$ holds, the following condition is needed.

(c3) $v \not\geq b$ for all $b \in P$.

As α is bounded by a constant, all these conditions can be checked in polynomial time for a given $v \in \{0,1\}^n$. This proves ADDITIONAL α-INTERIOR $\in$ NP.

To prove the completeness, we reduce the following NP-complete problem [11] to ADDITIONAL α-INTERIOR.

Problem HYPERGRAPH 2-COLORING

Input: A 3-uniform hypergraph $\mathcal{H} = (V, H)$ such that $V = \{1, 2, \ldots, n\}$ and $H = \{E_1, E_2, \ldots, E_m\}$, where $E_p \subseteq V$ and $|E_p| = 3$ for all p.

Question: Is $\mathcal{H}$ 2-colorable; i.e., does there exist a partition $(C, V \setminus C)$ of V such that $C \cap E_p \neq \emptyset$ and $(V \setminus C) \cap E_p \neq 0$ for all $E_p \in H$?

Without loss of generality, we assume that, for every pair of $l_1, l_2 \in V$, there exists an $E_p \in H$ satisfying $E_p \cap \{l_1, l_2\} = \emptyset$. This does not affect the NP-hardness of the problem.

Let k be the smallest integer satisfying $\binom{k}{\alpha} \geq m$, and let $V' = \{n+1, n+2, \ldots, n+k\}$. Construct a positive function f of $n+k$ variables from $V \cup V' = \{1, 2, \ldots, n+k\}$ as follows.

$$\min T(f) = \mathrm{MinSet}\{Q \cup (\cup_{q=1}^{\binom{k}{\alpha}} R_q) \cup U\}, \tag{3.3}$$

where

$$\begin{aligned}
&Q = \{w \mid ON(w) = E_p \cup T_q,\ p = 1, 2, \ldots, m, q = 1, 2, \ldots, \tbinom{k}{\alpha}\},\\
&R_q = \begin{cases} \{w \mid ON(w) = \{l\} \cup T_q,\ \text{for } l \in E_q\} & \text{for } q = 1, 2, \ldots, m-1,\\ \{w \mid ON(w) = \{l\} \cup T_q,\ \text{for } l \in E_m\} & \text{for } q = m, \ldots, \tbinom{k}{\alpha}, \end{cases}\\
&U = \{w \mid ON(w) = T'_q,\ q = 1, 2, \ldots, \tbinom{k}{\alpha-1}\},
\end{aligned}$$

where $T_1, T_2, \ldots, T_{\binom{k}{\alpha}}$ are all the subsets of V' with $|T_i| = k - \alpha$, and $T'_1, T'_2, \ldots, T'_{\binom{k}{\alpha-1}}$ are all the subsets of V' with $|T'_i| = k - \alpha + 1$. The corresponding instance of ADDITIONAL α-INTERIOR is defined by $\min T(f)$ and

$$P = \{w^{(p)} \mid ON(w^{(p)}) = E_p \cup V',\ p = 1, 2, \ldots, m\}. \tag{3.4}$$

Let us first show that $P \subseteq \min T(\sigma_{-\alpha}(f))$; i.e., (3.3) and (3.4) in fact give a problem instance. We claim that all $w^{(p)} \in P$ satisfy condition (c1), i.e., $w^{(p)} \in T(\sigma_{-\alpha}(f))$. Assume that

$$ON(w^{(p)}) = E_p \cup V'. \tag{3.5}$$

If $S \subseteq ON(w^{(p)})$ with $|S| = \alpha$ satisfies $|S \cap E_p| = 0$ (i.e., $S \subseteq V'$), then $u = w^{(p)} - \sum_{l \in S} e^{(l)} \in Q$ holds, and hence $u \in T(f)$. Otherwise (i.e., $|S \cap E_p| \geq 1$), we have a vector $w \in U$ such that $w \leq u$. This again implies $u \in T(f)$. Hence $P \subseteq T(\sigma_{-\alpha}(f))$.

We then show that all $w^{(p)} \in P$ satisfy condition (c2); i.e., $w^{(p)}$ is minimal in $T(\sigma_{-\alpha}(f))$, which completes the claim. Taking a $w^{(p)} \in P$, we consider two cases: (1) $i \in V'$ and (2) $i \in E_p$. In case (1), take an $S \subseteq V' \setminus \{i\}$ with $|S| = \alpha$ arbitrarily (clearly, such an S satisfies $S \subseteq ON(w^{(p)} - e^{(i)})$), and let $u = w^{(p)} - e^{(i)} - \sum_{l \in S} e^{(l)}$. Note that u satisfies $|ON(u) \cap V'| = k - \alpha - 1$, but $|ON(x) \cap V'| \geq k - \alpha$ holds for all $x \in \min T(f)$. These imply that no $x \in \min T(f)$ satisfies $x \leq u$, and hence $u \in F(\sigma_{-\alpha}(f))$. This result also shows that all $v \in \min T(\sigma_{-\alpha}(f))$ satisfy

(c4) $ON(v) \supseteq V'$.

Next, in case (2), let $E_p \setminus \{i\} = \{l_1, l_2\}$. By assumption on $\mathcal{H}$, some $E_q \in H$ satisfies $E_q \cap \{l_1, l_2\} = \emptyset$. For such a q, let $S = V' \setminus T_q$ and $u = w^{(p)} - e^{(i)} - \sum_{l \in S} e^{(l)}$. Then, any $x \in \min T(f)$ with $x \leq u$ (if exists) must satisfy $|ON(x) \cap V| \leq 2$ and $ON(x) \cap V' \subseteq T_q$, and hence such an x must belong to R_q. However, since $E_q \cap \{l_1, l_2\} = 0$, no $x \in R_q$ satisfies $x \leq u$. Hence $u \in F(\sigma_{-\alpha}(f))$. Therefore, we conclude that $w^{(p)}$ satisfies (c2).

We next show that $\min T(\sigma_{-\alpha}(f)) \setminus P \neq \emptyset$ if and only if $\mathcal{H}$ is 2-colorable, which completes the proof. For the only-if part, let $v \in \min T(\sigma_{-\alpha}(f)) \setminus P$. Then we claim that $(C, V \setminus C)$ is a 2-coloring of $\mathcal{H}$, where $C = V \cap ON(v)$. By conditions (c3), (c4) and (3.4),

$$C \not\supseteq E_p \text{ (i.e., } (V \setminus C) \cap E_p \neq \emptyset) \tag{3.6}$$

holds for all $E_p \in \mathcal{H}$. Furthermore by (c1), $u^{(p)} = v - \sum_{l \in V' \setminus T_p} e^{(l)} \in T(f)$ holds for every T_p; i.e., there exists an $x^{(p)} \in \min T(f)$ such that $x^{(p)} \leq u^{(p)}$. By (3.6) and (3.3), such an $x^{(p)}$ must belong to R_p, i.e.,

$$C \cap E_p \neq \emptyset. \tag{3.7}$$

These (3.6) and (3.7) prove the only-if part.

For the if part, let $(C, V \setminus C)$ be a 2-coloring of $\mathcal{H}$, and let v be the vector defined by

$$ON(v) = C \cup V'.$$

We first show that this v satisfies condition (c1). Let $u = v - \sum_{l \in S} e^{(l)}$, where $S \subseteq ON(v)$ with $|S| = \alpha$. We consider two cases $S \not\subseteq V'$ and $S \subseteq V'$. If $S \not\subseteq V'$, then $|ON(u) \cap V'| \geq k - \alpha + 1$ holds and some $w \in U$ satisfies $w \leq u$; i.e., $u \in T(f)$. Otherwise (i.e., $S \subseteq V'$), $ON(u) \cap V' = T_q$ holds for some T_q. Since $(C, V \setminus C)$ is a 2-coloring of $\mathcal{H}$, $u \geq w$ holds for some $w \in R_q$ and hence $u \in T(f)$. The v also satisfies (c3) by the definition of P. By (c1) and (c3), there is a vector $v' \leq v$ such that $v' \in \min T(\sigma_{-\alpha}(f)) \setminus P$. Hence the if part holds.

Finally, since $(n + k)(|\min T(f)| + |P|)$ is a polynomial in $|V| + |H|$, the above reduction can be performed in polynomial time. □

Based on this, we obtain the next result.

Theorem 3.1 There is no polynomial total time algorithm for α-INTERIOR, even if α is bounded by a constant, unless P=NP.

Proof. Assume that there is a polynomial total time algorithm A for α-INTERIOR, with polynomial running time $p(I, O)$, where I is the input length and O the output length. In order to solve ADDITIONAL α-EXTERIOR, execute A until either (i) it halts or (ii) time $p(I, |P|)$ is reached, where P is the input to ADDITIONAL α-INTERIOR. In case of (i), if A has output exactly $|P|$ vectors (resp., more than $|P|$ vectors), then output "yes" (resp., "no"). In case of (ii), output "no", since it implies that $|\min T(\sigma_{-\alpha}(f))| > |P|$. Therefore, ADDITIONAL α-INTERIOR can be solved in polynomial time in the input length $I + |P|$, which contradicts Lemma 3.1 unless P=NP. □

4 h-term DNF functions

A positive function f is called an h-term DNF if it has a DNF with at most h terms. This is equivalent to the condition

$$|\min T(f)| \leq h.$$

For two vectors v and w, let $v \vee w$ denote the componentwise OR (i.e., union) of v and w. For example, if $v = (1100)$ and $w = (0110)$, then $v \vee w = (1110)$. We denote by $\bigvee_{v \in R} v$ the union of all vectors in R.

4.1 Problem α-INTERIOR($\mathcal{C}_{h\text{-}term}$)

Lemma 4.1 Let f be a positive function and α be a nonnegative integer. Then every vector $v \in \min T(\sigma_{-\alpha}(f))$ can be represented by $v = \bigvee_{w \in R} w$ for some $R \subseteq \min T(f)$.

Proof. We prove the lemma by induction on α. Since $\sigma_0(f) = f$, it clearly holds for $\alpha = 0$. Assuming that it holds for $\alpha = k$, we consider the case of $\alpha = k + 1$.

Let $v \in \min T(\sigma_{-k-1}(f))$. Since $\sigma_{-k-1}(f) = \sigma_{-1}(\sigma_{-k}(f))$ by property (iii) in Section 1, the definition of $\sigma_{-1}(\cdot)$ implies that $v - e^{(i)} \in T(\sigma_{-k}(f))$ holds for every $i \in ON(v)$, i.e., there exists a vector $w^{(i)} \in \min T(\sigma_{-k}(f))$ such that $w^{(i)} \leq v - e^{(i)}$. Let $u = \bigvee_{i \in ON(v)} w^{(i)}$. We show that $u = v$ holds. First $u \leq v$ is obvious from its definition. If $u < v$, then it follows from $w^{(i)} \leq u$ and $w^{(i)}_i = 0$ that

$$w^{(i)} \leq u - e^{(i)} \text{ for all } i \in ON(u)\,(\subseteq ON(v)).$$

Definition (1.1) implies $u \in T(\sigma_{-1}(\sigma_{-k}(f))) = T(\sigma_{-k-1}(f))$, contradicting the assumption that $v \in \min T(\sigma_{-1}(f))$. Thus we have $v = u$. Now, by the induction hypothesis, $w^{(i)}$ can be represented by

$$w^{(i)} = \bigvee_{w \in R^{(i)}} w$$

for some $R^{(i)} \subseteq \min T(f)$. Then $v = \bigvee_{w \in R} w$ holds for $R = \bigcup_i R^{(i)}$. This proves the lemma. □

Let

$$Q = \{v \mid v = \bigvee_{w \in R} w \text{ for some } R \subseteq \min T(f)\}. \tag{4.8}$$

The above lemma suggests the following procedure for solving α-INTERIOR($\mathcal{C}_{h\text{-}term}$).

Step 1: Compute the set Q.

Step 2: For each $v \in Q$, check if $v \in \min T(\sigma_{-\alpha}(f))$ holds (by conditions (c1) and (c2) given in the proof of Lemma 3.1). □

Note that, $|Q| \leq 2^{|\min T(f)|}$ holds. Therefore $|Q|$ is polynomial in n if $|\min T(f)| = O(\log n)$. Even in this case, Step 2 of checking conditions (c1) and (c2) cannot be directly executed in polynomial time, if α is not bounded by a constant. In order to overcome this, we make use of the property $\sigma_{-\alpha-1}(f) = \sigma_{-1}(\sigma_{-\alpha}(f))$ of (iii) in Section 1. Namely, we first check if $v \in \min T(\sigma_{-1}(f))$ holds for all $v \in Q$. By Lemma 4.1, we can obtain $\min T(\sigma_{-1}(f))$ by this computation. Then we eliminate $\min T(\sigma_{-1}(f))$ and $F(\sigma_{-1}(f))$ from Q, since such vectors have no chance to belong to $\min T(\sigma_{-\alpha}(f))$ for $\alpha \geq 2$. Note that all these can be done in polynomial time. We then check if $v \in \min T(\sigma_{-1}(\sigma_{-1}(f)))$ $(= \min T(\sigma_{-2}(f)))$ holds for all $v \in Q$. Since $\min T(\sigma_{-1}(f))$ is already known, this also can be done in polynomial time. This gives $\min T(\sigma_{-2}(f))$. By applying this argument repeatedly, we can eventually compute $\min T(\sigma_{-\alpha}(f))$.

Algorithm INT
Input: $\min T(f)$ and a positive integer α, where f is a positive function.
Output: $\min T(\sigma_{-\alpha}(f))$.

Step 1: $\beta := 0, Q_0 := \min T(f), Q_l := \emptyset$, for $l = 1, 2, \ldots, \alpha$ (Q_l will contain $\min T(\sigma_{-l}(f))$ upon the completion of computation), and compute the set Q of (4.8).

Step 2: For each $v \in Q$, if $v \in \min T(\sigma_{-1}(\sigma_{-\beta}(f)))$ holds, then let $Q_{\beta+1} := Q_{\beta+1} \cup \{v\}$ and $Q := Q \setminus \{v\}$. If $v \in F(\sigma_{-1}(\sigma_{-\beta}(f)))$, then $Q := Q \setminus \{v\}$.

Step 3: $\beta := \beta + 1$. If $\beta = \alpha$, then output Q_β as $\min T(\sigma_{-\alpha}(f))$, and halt. Otherwise return to Step 2. □

Example 4.1 Given a positive function f defined by $\min T(f) = \{v^{(1)}, v^{(2)}, v^{(3)}, v^{(4)}\}$, where

$$v^{(1)} = (110000),\ v^{(2)} = (001100),\ v^{(3)} = (100010),\ v^{(4)} = (010010).$$

and $\alpha = 2$, algorithm INT runs as follows:

1 **(Step 1):** $\beta := 0, Q_0 := \min T(f), Q_l := \emptyset$ for $l = 1, 2\,(= \alpha)$, and $Q = \{v^{(1)}, v^{(2)}, v^{(3)}, v^{(4)}, v^{(1)} \vee v^{(2)}, v^{(1)} \vee v^{(3)}, \ldots, v^{(1)} \vee v^{(2)} \vee v^{(3)} \vee v^{(4)}\}$.

2 **(Steps 2 and 3 with $\beta = 0$):** For each $v \in Q$, check if $v \in \min T(\sigma_{-1}(f))$ holds by conditions (c1) and (c2) in the proof of Lemma 3.1. We obtain $Q_1 = \{v^{(1)} \vee v^{(2)}, v^{(1)} \vee v^{(3)}\ (= v^{(1)} \vee v^{(4)} = v^{(3)} \vee v^{(4)} = v^{(1)} \vee v^{(3)} \vee v^{(4)}), v^{(2)} \vee v^{(3)}, v^{(2)} \vee v^{(4)}\}$, where

$$v^{(1)} \vee v^{(2)} = (111100), \quad v^{(1)} \vee v^{(3)} = (110010),$$
$$v^{(2)} \vee v^{(3)} = (101110), \quad v^{(2)} \vee v^{(4)} = (011110),$$

For example, $v^{(1)} \vee v^{(2)} \in \min T(\sigma_{-1}(f))$ holds, since all $w^{(i)} = v^{(1)} \vee v^{(2)} - e^{(i)}$ for all $i \in ON(v^{(1)} \vee v^{(2)})$ (i.e., $(011100), (101100), (110100)$, or (111000)) belong to $\min T(f)$ as easily checked (condition (c1)), and for every $w^{(i)}$, there is an $e^{(l)}$ such that $w^{(i)} - e^{(l)} \in F(f)$ (condition (c2)). Then $Q_1 = \min T(\sigma_{-1}(f))$ holds. Moreover, we update Q as $Q = \{v^{(1)} \vee v^{(2)} \vee v^{(3)} (= v^{(1)} \vee v^{(2)} \vee v^{(4)} = v^{(2)} \vee v^{(3)} \vee v^{(4)} = v^{(1)} \vee v^{(2)} \vee v^{(3)} \vee v^{(4)})\}$, and let $\beta := 1$.

3 **(Steps 2 and 3 with $\beta = 1$):** For each $v \in Q$, check if $v \in \min T(\sigma_{-1}(\sigma_{-1}(f)))$ holds by conditions (c1) and (c2). Then we obtain $Q_2 = \{v^{(1)} \vee v^{(2)} \vee v^{(3)}\}$, where

$$v^{(1)} \vee v^{(2)} \vee v^{(3)} = (111110).$$

INT outputs $Q_2 = \min T(\sigma_{-2}(f))$, and halts. □

Note that $Q_\beta = \min T(\sigma_{-\beta}(f))$ holds for all β upon the completion of Algorithm INT. Thus, Algorithm INT for $\alpha = n$ can generate all interior functions $\sigma_{-\beta}(f)$ for $\beta = 1, 2, \ldots, n, \ldots$, since $\sigma_{-\beta}(f) = \perp$ holds for all $\beta \geq n$.

Theorem 4.1 Algorithm INT solves problem α-INTERIOR in $O(\alpha n^3 2^{2|\min T(f)|})$ time.

Proof. The correctness of INT follows from the above discussion. As for its time complexity, Step 1 can be done in $O(|\min T(f)| n 2^{|\min T(f)|})$ time, since $|Q| = 2^{|\min T(f)|} - 1$ and Q can be computed in this amount of time (the rest of Step 1 can be executed in $O(\alpha)$ time). In Step 2, we check if $v \in \min T(\sigma_{-1}(\sigma_{-\beta}(f)))$ holds and if $v \in F(\sigma_{-1}(\sigma_{-\beta}(f)))$ holds for every $v \in Q$. By conditions (c1) and (c2) in the proof of Lemma 3.1, these can be done by checking if $v - e^{(i)} \in T(\sigma_{-\beta}(f))$ for all $i \in ON(v)$ (condition (c1)), and by checking for every $i \in ON(v)$ if there exists $l \in ON(v)$ such that $v - e^{(i)} - e^{(l)} \in F(\sigma_{-\beta}(f))$ (condition (c2)). Since checking $u \in T(\sigma_{-\beta}(f))$ and $u \in F(\sigma_{-\beta}(f))$ can be done in $n|\min T(\sigma_{-\beta}(f))|$ $(\leq n|Q|)$ time, each iteration of Step 3 can be executed in

$$O(2^{|\min T(f)|} \times n^2 \times n 2^{|\min T(f)|}) = O(n^3 2^{2|\min T(f)|})$$

time. Since we have α iterations of Steps 1 - 3, the total time is bounded from above by $O(\alpha n^3 2^{2|\min T(f)|})$. □

Corollary 4.1 Problem α-INTERIOR can be solved in polynomial time, if f is an $O(\log n)$-term DNF function. □

4.2 Problem α-EXTERIOR($\mathcal{C}_{h\text{-}term}$)

We investigate exterior functions of h-term DNFs. Recall that, in general, Problem α-EXTERIOR is intractable, unless P=NP [13]. Although Theorem 1.1(ii) characterize $\min T(\sigma_\alpha(f))$, it does not immediately give a polynomial total time algorithm, even f is restricted to h-term DNF, as the following example indicates.

Example 4.2 Let n be an even positive integer, and let f be a positive function defined by $\min T(f) = \{v^{(1)}, v^{(2)}, v^{(3)}\}$, where

$$v^{(1)} = (\underbrace{11\ldots1}_{n/2+1}00\ldots0), \quad v^{(2)} = (100\ldots0\underbrace{11\ldots1}_{n/2}), \quad v^{(3)} = (0\underbrace{11\ldots1}_{n-1}).$$

This f is clearly a 3-term DNF. Then consider Problem $n/2$-EXTERIOR. By definition, it is not difficult to see that

$$\min T(\sigma_{n/2}(f)) = \{e^{(i)} \mid i = 1, 2, \ldots, n\}$$

holds. However, if Theorem 1.1 (ii) is directly applied to this case, we have the following intermediate set

$$\{v^{(i)} - \sum_{j\in S} e^{(j)} \mid i = 1, 2, 3, S \subseteq ON(v^i), |S| = n/2\},$$

which is exponentially large in n. This says that a direct application of Theorem 1.1 (ii) does not lead to an efficient algorithm for Problem α-EXTERIOR($\mathcal{C}_{h\text{-}term}$). □

In this subsection, however, we develop a polynomial delay algorithm to compute $\min T(\sigma_\alpha(f))$, assumming that the input f is an h-term DNF with a fixed h. Let

$$\min T(f) = \{v^{(1)}, v^{(2)}, \ldots, v^{(h)}\},$$

where we assume $|\min T(f)| = h$ and

$$|ON(v^{(i)})| \leq |ON(v^{(i+1)})| \quad \text{for } i = 1, 2, \ldots, h-1, \tag{4.9}$$

without loss of generality. We first check if $|ON(v^{(1)})| \leq \alpha$. If so, we can conclude $\sigma_\alpha(f) = \top$, and halt. Otherwise (i.e., $|ON(v^{(1)})| > \alpha$), we output all vectors,

$$v_S^{(1)} = v^{(1)} - \sum_{j\in S} e^{(j)}, \ S \subseteq ON(v^{(1)}) \text{ and } |S| = \alpha. \tag{4.10}$$

Since $|ON(v^{(1)})|$ is the smallest among all $|ON(v^{(i)})|$ of $v^{(i)} \in \min T(f)$ by (4.9), it follows from Theorem 1.1(ii) that all vectors $v_S^{(1)}$ of (4.10) are contained in $\min T(\sigma_\alpha(f))$.

In general, after processing $v^{(i)}$ for $i = 1, 2, \ldots, k-1$, we move to $v^{(k)}$ and output all vectors $v_S^{(k)}$ such that

$$v_S^{(k)} = v^{(k)} - \sum_{j \in S} e^{(j)}, \text{ such that } S \subseteq ON(v^{(k)}) \text{ and } |S| = \alpha, \tag{4.11}$$

for which there is no $v_{S'}^{(i)}$ such that $i \leq k-1$ and $v_{S'}^{(i)} \leq v_S^{(k)}$. Clearly, $v_S^{(k)} \in \min T(\sigma_\alpha(f))$ holds. Now, let Q_k denote the set of all vectors output from $v^{(1)}, v^{(2)}, \ldots, v^{(k)}$ in the above procedure:

$$\begin{aligned} Q_0 &= \emptyset, \\ Q_k &= \text{MinSet}\{a - \sum_{j \in S} e^{(j)} \mid a \in \{v^{(1)}, v^{(2)}, \ldots, v^{(k)}\}, S \subseteq ON(a), |S| = \alpha\}, \ k = 1, 2, \ldots, h. \end{aligned}$$

As explained above, we have $Q_k \subseteq \min T(\sigma_\alpha(f))$ for all $k \leq h$ and $Q_h = \min T(\sigma_\alpha(f))$. Hence, this algorithm solves α-EXTERIOR for positive h-term DNF functions.

In iteration k of the above computation, we generate all vectors $v_S^{(k)} \in Q_k \setminus Q_{k-1}$. However since $|Q_{k-1}|$ and the number of all $v_S^{(k)}$ of (4.11) can both be $O(n^\alpha)$ (but set $Q_k \setminus Q_{k-1}$ may be small, even empty), this computation is not polynomial time in the input and output size if it is done naively. Therefore we now consider how to generate $v_S^{(k)} \in Q_k \setminus Q_{k-1}$ more efficiently.

For a vector v and a set $I \subseteq \{1, 2, \ldots, n\}$, let $v[I]$ denote the projection of v on I; e.g., if $v = (1100)$ and $I = \{2, 3\}$, then $v[I] = (10)$. Let, for k and i with $k > i$,

$$Q'_{k,i} = \{v_{S'}^{(i)}[ON(v^{(k)})] \mid v_{S'}^{(i)} \in Q_{k-1}, v_{S'}^{(i)} < v^{(k)}\},$$

$$Q'_k = \bigcup_{i \leq k-1} Q'_{k,i},$$

and let $g_{k,i}$ (resp., g_k) be the positive function defined by $\min T(g_{k,i}) = Q'_{k,i}$ (resp., $\min T(g_k) = Q'_k$), where the variable set of $g_{k,i}$ (resp., g_k) is $ON(v^{(k)})$. For simplicity we denote

$$a_S = v_S^{(k)}[ON(v^{(k)})]$$

for $v_S^{(k)}$ of (4.11). These a_S satisfy $|OFF(a_S)| = \alpha$. We note that $v_S^{(k)} \in Q_k \setminus Q_{k-1}$ of (4.11) has no $b \in Q'_k$ such that $b \leq a_S$, which is equivalent to saying that $v_S^{(k)} \in Q_k \setminus Q_{k-1}$ satisfies $a_S \in F(g_k)$. Since each a_S can be extended to $v_S^{(k)}$ by adding 0's in the components $OFF(v^{(k)})$, there is an one-to-one correspondence between $v_S^{(k)}$ and a_S. Hence computing all

$$a_S \in F(g_k), \text{ such that } S \subseteq ON(v^{(k)}) \text{ and } |S| = \alpha \tag{4.12}$$

means to compute all $v_S^{(k)} \in Q_k \setminus Q_{k-1}$.

In order to compute all a_S of (4.12), we next characterize g_k. Since $F(g_k) = \bigcap_{i=1}^{k-1} F(g_{k,i})$, any $a_S \in F(g_k)$ satisfies $g_{k,i}(a_S) = 0$ for all $i \leq k-1$. For $g_{k,i}$, let

$$\begin{aligned} a^{(k,i)} &= (v^{(k)} \wedge v^{(i)})[ON(v^{(k)})] \\ \beta_{k,i} &= \alpha - |ON(v^{(i)}) \setminus ON(v^{(k)})|. \end{aligned}$$

Note that $v^{(k)} \wedge v^{(i)}$ is the vector in set $\{u \mid u < v^{(k)}\}$ nearest to $v^{(i)}$. If $|ON(v^{(i)}) \setminus ON(v^{(k)})| > \alpha$, then no $v^{(i)}_{S'} \in Q_{k-1}$ satisfies $v^{(i)}_{S'} < v^{(k)}$. Otherwise all $v^{(i)}_{S'}$ with $v^{(i)}_{S'} < v^{(k)}$ can be written as $v^{(i)}_{S'} = v^{(i)} - \sum_{j \in S'} e^{(j)} = v^{(i)} \wedge v^{(k)} - \sum_{j \in S' \cap ON(v^{(k)})} e^{(j)}$. Note that $|S' \cap ON(v^{(k)})| = \beta_{k,i}$ must hold in this case. Now if a vector a_S of (4.12) satisfies $g_{k,i}(a_S) = 0$, there is no $b_{S'} = v^{(i)}_{S'}[ON(v^{(k)})]$ such that $b_{S'} \leq a_S$; i.e., S satisfies $ON(a_S) \not\supseteq ON(b_{S'}) = ON(a^{(k,i)}) \setminus (S' \cap ON(v^{(k)}))$ for all $S' \subseteq ON(v^{(k)})$ with $|S' \cap ON(v^{(k)})| = \beta_{k,i}$. This is equivalent to the condition $|ON(a^{(k,i)}) \cap OFF(a_S)| \geq \beta_{k,i} + 1$. Therefore the computation of all S of (4.12) can be done by solving the following integer program over the 0-1 variables y_j corresponding to $j \in ON(v^{(k)})$, where $y_j = 1$ denotes $j \in S$:

$$\sum_{j \in ON(a^{(k,i)})} y_j \quad \geq \quad \beta_{k,i} + 1, \quad i = 1, 2, \ldots, k-1. \tag{4.13}$$

As S satisfies $|S| = \alpha$, we also have condition

$$\sum_{j \in ON(v^{(k)})} y_j \quad = \quad \alpha. \tag{4.14}$$

This integer program can be solved by dynamic programming as follows. For notational simplicity, assume $ON(v^{(k)}) = \{1, 2, \ldots, L\}$, where $L = |ON(v^{(k)})|$, and let $J_l = \{1, 2, \ldots, l\}$. Define the states of dynamic programming $Y_l(\gamma)$ for $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_{k-1}) \in \Gamma$, where $\Gamma = \{\gamma \mid \gamma_0 \in \{0, 1, \ldots, \alpha\}$ and $\gamma_i \in \{0, 1, \ldots, \beta_{k,i} + 1\}, i = 1, 2, \ldots, k-1\}$, by

$$Y_l(\gamma) = \begin{cases} 1 & \text{if there exists } (y_1, y_2, \ldots, y_l) \in \{0,1\}^l \text{ such that} \\ & \quad \textstyle\sum_{j \in J_l} y_j = \gamma_0 \qquad (\textstyle\sum_{j \in J_l} y_j = \alpha \text{ if } \gamma_0 = \alpha) \text{ and} \\ & \quad \textstyle\sum_{j \in ON(a^{(k,i)}) \cap J_l} y_j \geq \gamma_i \quad (\textstyle\sum_{j \in ON(a^{(k,j)}) \cap J_l} y_j \geq \beta_{k,i} + 1 \text{ if } \gamma_i = \beta_{k,i} + 1). \\ 0 & \text{otherwise.} \end{cases}$$

Then $Y_l(\gamma)$ can be computed by the following dynamic programming recursion:

$$Y_1(\gamma) \quad = \quad \begin{cases} 1 & \text{if } \gamma = (00\ldots0) \text{ or } \gamma = (1, |ON(a^{(k,1)}) \cap \{1\}|, \ldots, |ON(a^{(k,k-1)}) \cap \{1\}|) \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{for } \gamma \in \Gamma,$$

$$Y_l(\gamma) \quad = \quad \begin{cases} 1 & \text{if } Y_{l-1}(\gamma) = 1 \text{ or } Y_{l-1}(\gamma') = 1, \text{ where} \\ & \gamma' = (\gamma_0 - 1, \gamma_1 - |ON(a^{(k,1)}) \cap \{l\}|, \ldots, \gamma_{k-1} - |ON(a^{(k,k-1)}) \cap \{l\}|) \\ 0 & \text{otherwise,} \end{cases}$$

$$\text{for } l = 2, 3, \ldots, k-1, \text{ and } \gamma \in \Gamma. \tag{4.15}$$

By the definition of the states, there is a solution y to integer program (4.13) and (4.14) if and only if $Y_{k-1}(\gamma^*) = 1$ holds for $\gamma^* = (\alpha, \beta_{k,1} + 1, \ldots, \beta_{k,k-1} + 1)$. Since $\alpha \leq n$ and $\beta_{k,i} < n$, the number of states in all stages $l = 1, 2, \ldots, k-1$ is bounded by $(k-1)|\Gamma| = O(kn^k) = O(kn^h)$. Therefore, (4.15) can be solved in $O(h^2 n^h)$ time, which is polynomial since h is bounded by a constant. Although we omit details, if $Y_{k-1}(\gamma^*) = 1$, all the solutions y to (4.13) and (4.14) can be recovered from the

recursion (4.15) in polynomial delay time by backtracking all the states $Y_l(\gamma) = 1$ which lead to $Y_{k-1}(\gamma^*) = 1$.

Since $a^{(k,i)}$ and $\beta_{k,i}$ can be computed in polynomial time, and the solutions y of (4.13) and (4.14) can be transformed into $v_S^{(k)} \in Q_k \setminus Q_{k-1}$, the above procedure gives a polynomial delay algorithm for problem α-EXTERIOR if the input f is an h-term DNF function, with a constant h. Formally, the algorithm can be written as follows.

Algorithm EXT (to solve problem α-EXTERIOR)
Input: $\min T(f)$ and an nonnegative integer α, where f is a positive function.
Output: $\min T(\sigma_\alpha(f))$.

Step 0: Arrange vectors in $\min T(f)$ so that (4.9) holds.

Step 1: If $|ON(v^{(1)})| \leq \alpha$, then output $\{(00\ldots00)\}$ (i.e., $\min T(\sigma_\alpha(f)) = \top$) and halt; else output set Q_1, i.e., all vectors $v_S^{(1)}$ of (4.10).

Step 2: For $k = 2, 3, \ldots, h$, output all vectors $v_S^{(k)} \in Q_k \setminus Q_{k-1}$ of (4.11) with $S = ON(y)$ for all solutions y of (4.13) and (4.14).

Example 4.3 Given a positive function f defined by $\min T(f) = \{v^{(1)}, v^{(2)}, v^{(3)}\}$, where

$$v^{(1)} = (001100101),\ v^{(2)} = (111100110),\ v^{(3)} = (111111000), \tag{4.16}$$

and $\alpha = 3$, Algorithm EXT proceeds as follows:

1: Since $\alpha = 3 \leq |ON(v^{(1)})| = 4$, EXT outputs set $Q_1 = \{(001000000), (000100000), (000000100), (000000001)\}$, by computing $v_S^{(1)}$ of (4.10).

2: We have $ON(v^{(1)}) \setminus ON(v^{(2)}) = \{9\}$, $\beta_{2,1} = 3 - 1 = 2$, $v^{(2)} \wedge v^{(1)} = (001100100)$, $a^{(2,1)} = (0011{*}{*}10{*})$, where the elements not in $ON(v^{(2)})$ are indicated by $*$. The linear inequalities (4.13) and (4.14) are written as:

$$\sum_{j \in ON(a^{(2,1)}) = \{3,4,7\}} y_j \ \geq\ \beta_{2,1} + 1 \ =\ 3, \qquad \sum_{j \in ON(v^{(2)}) = \{1,2,3,4,7,8\}} y_j \ =\ \alpha \ =\ 3, \tag{4.17}$$

which has only one solution $(0011{*}{*}10{*})$. By this solution and (4.11), EXT produces $Q_2 \setminus Q_1 = \{(110000010)\}$.

3: We have $ON(v^{(1)}) \setminus ON(v^{(3)}) = \{7, 9\}$, $\beta_{3,1} = 3 - 2 = 1$, $v^{(3)} \wedge v^{(1)} = (001100000)$, $a^{(3,1)} = (001100{*}{*})$, and, $ON(v^{(2)}) \setminus ON(v^{(3)}) = \{7, 8\}$, $\beta_{3,2} = 3 - 2 = 1$, $v^{(3)} \wedge v^{(2)} = (111100000)$, $a^{(3,2)} = (111100{*}{*})$. The linear inequalities (4.13) and (4.14) now become:

$$\sum_{j \in ON(a^{(3,1)}) = \{3,4\}} y_j \ \geq\ \beta_{3,1} + 1 \ =\ 2, \qquad \sum_{j \in ON(a^{(3,2)}) = \{1,2,3,4\}} y_j \ \geq\ \beta_{3,2} + 1 \ =\ 2,$$
$$\sum_{j \in ON(v^{(3)}) = \{1,2,3,4,5,6\}} y_j \ =\ \alpha \ =\ 3. \tag{4.18}$$

The set of solutions to (4.18) is $\{(101100***), (011100***), (001110***), (001101***)\}$. Then by (4.11), EXT constructs $Q_3 \setminus Q_2 = \{(010011000), (100011000), (110001000), (110010000)\}$, and halts. We have

$$\begin{aligned} \min T(\sigma_3(f)) &= Q_3 = \{(00100000), (00010000), (00000100), (00000001), \\ &\quad (110000010), (010011000), (100011000), (110001000), (110010000)\}. \end{aligned}$$

□

Theorem 4.2 Problem α-EXTERIOR for positive h-term DNF functions, where h is a constant, can be solved in polynomial delay time. □

5 k-DNF functions

A positive functions f is called k-DNF if

$$|ON(v)| \leq k$$

holds for all $v \in \min T(f)$. Let us start with problem α-EXTERIOR. If k is general, problem α-EXTERIOR is intractable unless P=NP, as stated in (II) in Section 2. However we can show below that it can be solved in polynomial time, if $k = O(\log n)$.

For a positive k-DNF function f, $\sigma_\alpha(f) = \top$ holds for all $\alpha \geq k$. Otherwise (i.e., $\alpha < k$), Theorem 1.1(ii) tells that $\min T(\sigma_\alpha(f))$ can be computed by considering at most 2^α (which is $O(n^c)$ if $\alpha < k = c \log n$ for a constant c) sets $S \subseteq ON(a)$ for each $a \in \min T(f)$. This establishes the next theorem.

Theorem 5.1 Let f be a positive $O(\log n)$-DNF function. Then problem α-EXTERIOR is polynomially solvable. □

We next consider the interior functions. It is known that problem α-INTERIOR is in general intractable, even if the input f is 2-DNF [14]. After showing several lemmas, we show that if k and α are both bounded by some constants, or $k = 1$, then α-INTERIOR becomes tractable.

Lemma 5.1 Let f be a positive function, and let u be a vector in $\{0,1\}^n$. The $\sigma_{-1}(f)(u) = 1$ holds if and only if $u \geq \bigvee_{w \in R} w$ holds for some nonempty $R \subseteq \min T(f)$ satisfying $\bigcap_{w \in R} ON(w) = \emptyset$.

Proof. Let us first show the only-if part. Assume that $u \geq v$ holds for some $v \in \min T(\sigma_{-1}(f))$. By Lemma 4.1, this v can be represented by $\bigvee_{w \in R} w$, for some $R \subseteq \min T(f)$. Assume that $\bigcap_{w \in R} ON(w) \neq \emptyset$ holds, and let $i \in \bigcap_{w \in R} ON(w)$. By the definition of $\sigma_{-1}(f)$, $v - e^{(i)} \in T(f)$ holds, implying that there exists a $u^{(i)} \in \min T(f)$ such that $u^{(i)} \leq v - e^{(i)}$. Then clearly, we have

$v = u^{(i)} \vee \bigvee_{w \in R} w$. But $i \notin ON(u^{(i)})$ implies that $i \notin \bigcap_{w \in R \cup \{u^{(i)}\}} ON(w)$. Applying this argument to all $i \in \bigcap_{w \in R} ON(w)$, we prove the only-if part.

For the if part, let us assume that $u \geq v$ holds some $v = \bigvee_{w \in R} w$ for which $R \subseteq \min T(f)$ and $\bigcap_{w \in R} ON(w) = \emptyset$ hold. We show $\sigma_{-1}(f)(v) = 1$, and this proves the if part by the positivity of $\sigma_{-1}(f)$. Since $\cap_{w \in R} ON(w) = \emptyset$, for each $i \in ON(v)$ there is a $w^{(i)} \in R$ such that $w_i^{(i)} = 0$. This means $w^{(i)} \leq v - e^{(i)}$, and hence $f(v - e^{(i)}) = 1$ for all $i \in ON(v)$; i.e., $\sigma_{-1}(f)(v) = 1$ by (1.1). $\Box$

This lemma immediately implies the following corollary.

Corollary 5.1 Let f be a positive function. Then every $v \in \min T(\sigma_{-1}(f))$ can be represented by $v = \bigvee_{w \in R} w$ for some $R \subseteq \min T(f)$ with $\bigcap_{w \in R} ON(w) = \emptyset$. $\Box$

The next lemma says that, for a positive k-DNF function f, no R in Lemma 5.1 and Corollary 5.1 is large.

Lemma 5.2 Let f be a positive k-DNF function, where $k \geq 2$. Then every $v \in \min T(\sigma_{-1}(f))$ has a representation $v = \bigvee_{w \in R} w$ for some $R \subseteq \min T(f)$ such that $|R| \leq k$.

Proof. Take a $v \in \min T(\sigma_{-1}(f))$. By corollary 5.1, we have $v = \bigvee_{w \in R} w \in \min T(\sigma_{-1}(f))$ for some $R \subseteq \min T(f)$ with $\bigcap_{w \in R} ON(w) = \emptyset$. Assume that $R = \{w^{(1)}, w^{(2)}, \ldots, w^{(l)}\}$ is minimal, i.e., no $R' \subset R$, which satisfies $v = \bigvee_{w \in R'} w \in \min T(\sigma_{-1}(f))$, has the property of $\bigcap_{w \in R'} ON(w) = \emptyset$. Let $R_j = R \setminus \{w^{(j)}\}$, $j = 1, 2, \ldots, l$. Then by the minimality of R, we can show that

$$S_j = \bigcap_{w \in R_j} ON(w) \neq \emptyset, \text{ for all } j.$$

Then we claim that $S_j \neq \emptyset$ holds for all j. If $S_j = \emptyset$ holds, then for each $i \in ON(v^{(j)})$, where $v^{(j)} = \bigvee_{w \in R_j} w$, there exists a $w \in R_j$ such that $w \leq v^{(j)} - e^{(i)}$. This implies $v^{(j)} \in T(\sigma_{-1}(f))$. Moreover, by the minimality of R, $v > v^{(j)}$ holds, which contradicts that $v \in \min T(\sigma_{-1}(f))$.

Note that $|ON(w^{(1)})| \leq k$ holds. Also $S_j \subseteq ON(w^{(1)})$ holds for all $j \neq 1$. If $l \geq k + 2$, then by the pigeonhole principle, we have

$$S_{j_1} \cap S_{j_2} \neq \emptyset \tag{5.19}$$

for some distinct $j_1, j_2 \neq 1$. Since $R_{j_1} \cup R_{j_2} = R$ by definition, this implies

$$\bigcap_{w \in R} ON(w) = \bigcap_{w \in R_{j_1} \cup R_{j_2}} ON(w) = S_{j_1} \cap S_{j_2} \neq \emptyset,$$

a contradiction. Thus $l \leq k + 1$ must hold.

Then consider the case of $l = k + 1$. Since (5.19) leads to the above contradiction, we have $S_{j_1} \cap S_{j_2} = \emptyset$ for all $j_1, j_2 \neq 1$. Since $\bigcup_{j=2}^{k+1} S_j \subseteq ON(w^{(1)})$ and $|ON(w^{(1)})| \leq k$, we have

$$\bigcup_{j=2}^{k+1} S_j = ON(w^{(1)}),$$

where $k \geq 2$ is assumed. This means that every $i \in ON(w^{(1)})$ is contained in exactly one S_j, $j \neq 1$. Thus, we have $v = \bigvee_{w \in R} w = \bigvee_{w \in R_1} w$, again a contradiction. This proves $l \leq k$. □

This lemma gives rise to an algorithm for computing $\sigma_{-1}(f)$ of a positive k-DNF function f. Namely, for each $R \subseteq \min T(f)$ with $|R| \leq k$, we check if

$$\bigvee_{w \in R} w \in \min T(\sigma_{-1}(f)). \tag{5.20}$$

There are $O(|\min T(f)|^k)$ such R, and we can check condition (5.20) in polynomial time by conditions (c1) and (c2) in the proof of Lemma 3.1. Therefore, if k is bounded by a constant, problem 1-INTERIOR is polynomially solvable.

Then, Lemma 5.2 tells that

$$|ON(v)| \leq k^2$$

holds for all $v \in \min T(\sigma_{-1}(f))$, i.e., $\sigma_{-1}(f)$ is k^2-DNF. Thus, decomposing $\sigma_{-\alpha}$ to $\sigma_{-1}(\sigma_{-1}(\ldots\sigma_{-1}(f)\ldots))$, similarly to Algorithm INT, we see that $|ON(v)| \leq k^{2^\alpha}$ holds for all $v \in \sigma_{-\alpha}(f)$, which establishes the following result.

Theorem 5.2 If both k and α are bounded by some constants, problem α-INTERIOR for k-DNF functions can be solved in polynomial time.

Before concluding this section, we consider positive 1-DNF functions, for which the next lemma holds corresponding to Lemma 5.2.

Lemma 5.3 Let f be a positive 1-DNF function. Then $v \in \min T(\sigma_{-\alpha}(f))$ holds if and only if $v = \bigvee_{w \in R} w$ holds for some $R \subseteq \min T(f)$ with $|R| = \alpha + 1$.

Proof. Let us first show the if part. Since f is 1-DNF and $f \neq \top$, each $w \in \min T(f)$ has the single element $w_i = 1$ and $|ON(v)| = |R|$ holds. Thus, if $|R| = \alpha + 1$, then we have $v \in \min T(\sigma_{-\alpha}(f))$ by conditions (c1) and (c2) in the proof of Lemma 3.1.

For the only-if part, recall that every $v \in \min T(\sigma_{-\alpha}(f))$ can be represented by $v = \bigvee_{w \in R} w$ for some $R \subseteq \min T(f)$, by Lemma 4.1. If $|R| = |ON(v)| \leq \alpha$, then we have $(00\ldots0) \in N_\alpha(v)$. However $f(00\ldots0) = 0$ by the assumption $f \neq \top$, contradicting $\sigma_{-\alpha}(f)(v) = 1$. On the other hand, if $|R| = |ON(v)| \geq \alpha + 2$ holds, consider any set $R' \subseteq R$ with $|R'| = \alpha + 1$. It follows from the if part that $u = \bigvee_{w \in R'} w$ satisfies $u \in \min T(\sigma_{-\alpha}(f))$. Since $u < v$, this contradicts the assumption $v \in \min T(\sigma_{-\alpha}(f))$. Hence $|R| = \alpha + 1$ holds. □

By this lemma, in order to compute $\min T(\sigma_{-\alpha}(f))$ of a positive 1-DNF function f, we only generate all subsets of $\min T(f)$ of size $\alpha + 1$.

Example 5.1 Given a positive 1-DNF function f as $\min T(f) = \{v^{(1)} = (100000), v^{(2)} = (010000), v^{(3)} = (001000), v^{(4)} = (000100)\}$, we obtain $\min T(\sigma_{-2}(f))$, by generating all subsets R of $\min T(f)$ where $|R| = 2 + 1$, i.e.,

$$v^{(1)} \vee v^{(2)} \vee v^{(3)} = (111000),$$
$$v^{(1)} \vee v^{(2)} \vee v^{(4)} = (110100),$$
$$v^{(1)} \vee v^{(3)} \vee v^{(4)} = (101100),$$
$$v^{(2)} \vee v^{(3)} \vee v^{(4)} = (011100).$$

□

Theorem 5.3 There is a polynomial delay algorithm for problem α-INTERIOR (where α is general), if the input f is 1-DNF. □

6 Conclusion

In this paper, we investigated problems α-INTERIOR and α-EXTERIOR, introduced in [14], from the view-point of their computational complexity. We first answered a question of α-INTERIOR, left open in [14], by showing that it has no polynomial total time algorithm even if α is bounded by a constant. We then considered classes of h-term DNF and k-DNF positive functions. Problems α-INTERIOR and α-EXTERIOR for positive h-term DNF functions with h bounded by a constant have (input) polynomial time and polynomial delay algorithms, respectively. As for positive k-DNF functions with $k = O(\log n)$, problem α-EXTERIOR can be solved in input polynomial time. Although problem α-INTERIOR has no polynomial total time algorithm for positive k-DNF functions with general α and fixed $k \geq 2$ [14], we gave polynomial delay algorithms for α-INTERIOR for two cases in which $k = 1$ holds, and α and k are both bounded by a constant.

Some problems remain for further work. One issue is considering interior and exterior functions for other interesting special classes. Another issue is considering a general (i.e., nonpositive) case.

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