# Weighted Max-Norm Estimate of Additive Schwarz Iteration Scheme for Solving Linear Complementarity Problems*

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**Abstract**

In this paper, we consider an algebraic additive Schwarz iteration scheme for solving the finite-dimensional linear complementarity problem that involves an M-matrix. The scheme contains some existing algorithms as special cases. We establish monotone convergence of the iteration scheme under appropriate conditions. Moreover, using the concept of weak regular splitting, we estimate weighted max-norm bounds for iteration errors; thereby we show that the sequence generated by the iteration scheme converges to the unique solution of the problem without any restriction on the initial point.

*Keywords:* algebraic additive Schwarz iteration, linear complementarity problem, monotone convergence, weighted max-norm

* The work of the first two authors was done while they were visiting Kyoto University.

# 1. Introduction

We consider the finite-dimensional linear complementarity problem (LCP) of finding an $x \in R^n$ such that

$$x \geq \phi, \quad Ax - F \geq 0, \quad (x-\phi)^T (Ax - F) = 0, \tag{1.1}$$

where $A \in R^{n\times n}$ is a given matrix, and $\phi$, $F \in R^n$ are given vectors. If $\phi = -\infty$, then problem (1.1) reduces to the system of linear equations

$$Ax = F. \tag{1.2}$$

In this paper, we assume that $A = (a_{ij})$ is an M-matrix [16, 20], i.e., it is a nonsingular matrix with non-positive off-diagonals and nonnegative inverse $A^{-1} \geq 0$, where the last inequality is understood to be component-wise. It is well-known that if $A$ is an M-matrix, then the LCP (1.1) has a unique solution. LCPs with an M-matrix have many practice applications in science and engineering. For example, it arises from discretizing a unilateral obstacle problem with an elliptic differential operator [9, 10]. Iterative methods constitute an important class of methods for solving LCPs. This class is particularly useful for solving large sparse problems. Recently, various Schwarz iterative algorithms for solving finite-dimensional variational inequalities as well as complementarity problems have been presented [1, 5, 9, 10, 11, 12, 13, 17, 19, 21]. This kind of methods are amenable to parallel computation. Moreover, the convergence rate will not be deteriorated with the refinement of the mesh when applied to discretized differential equations. Numerical experiments have shown that the latter advantage is still maintained when the methods are used to solve discretized variational inequalities with an elliptic differential operator [19, 21]. Theoretically, there are generally two ways of studying convergence of Schwarz method for solving LCPs. One is to prove that the method generates a minimizing sequence for some objective function. In this case, matrix $A$ is often supposed to be symmetric and positive definite. However, so far, no convergence rate has been given by such an approach. The other way is to prove that the method produces a monotone sequence starting from a super-solution or a lower-solution of the problem. Convergence theorems established in the latter way are generally based on the assumption that matrix $A$ is an M-matrix. Moreover, iterative error bounds have been obtained by using spectral norm [9, 10] or max norm [21].

This paper will consider an additive Schwarz iteration scheme for solving the LCP (1.1). To motivate, we first briefly review a general additive Schwarz method for solving the linear equation (1.2). The concept of additive Schwarz iteration for solving (1.2) was introduced by Dryja and Widlund [2]. See also [3, 4, 8, 13, 18] and the extensive bibliography therein. To be specific, let $V_i \subset R^n$, $i = 1, 2, \ldots, m$, be subspaces such that

$$\sum_{i=1}^{m} V_i \equiv \{v \in V : v = v_1 + \ldots + v_m, v_i \in V_i \ (i = 1, \ldots, m)\} = R^n. \tag{1.3}$$

That is, the bases of the subspaces $V_i$ altogether span the whole space. Then one step of an additive Schwarz iteration scheme consists of the following process: Restrict the current residual and solve the local problem on each subspace $V_i$, prolongate the approximations of the errors, and add the errors to the correction. Details will be given in the next section.

The purpose of this paper is to apply additive Schwarz iteration scheme to solve the LCP (1.1). The scheme is an extension of the additive Schwarz iteration scheme for solving the linear equation (1.2), which was proposed by Frommer and Szyld [4]. It also contains some existing additive Schwarz algorithms for solving LCPs as special cases [9, 10, 13]. We show that the proposed method generates a monotone sequence of iterates if the initial point is a super-solution of the problem. Moreover, without any restriction on the initial point, we obtain weighted max-norm bounds for iteration errors and establish convergence of the generated sequence to the solution of the problem.

The paper is organized as follows. In Section 2 we propose an additive Schwarz iteration scheme for solving (1.1). In Section 3, we give some basic properties of the proposed iterartion scheme. In Sections 4, we show monotone convergence of the scheme. In Section 5, we estimate the weighted max-norm bound for iteration errors and then establish global convergence of the scheme.

## 2. Additive Schwarz Iteration Scheme

In this section, we propose an additive Schwarz iteration scheme for solving (1.2). Let $V_i$ be subspaces of $R^n$ satisfying (1.3) and $n_i = \dim(V_i)$ be the dimension of subspace $V_i$, $i = 1, 2, \ldots, m$. We consider both overlapping subdomains and nonoverlapping subdomains, which correspond to the cases $\sum_{i=1}^{m} n_i > n$ and $\sum_{i=1}^{m} n_i = n$, respectively. For simplicity, we identify $V_i$ by $R^{n_i}$. Let $R_i : R^n \to R^{n_i}$ be the restriction operator. In our context, $R_i$ is an $n \times n_i$ matrix with $\operatorname{rank}(R_i) = n_i$. Its transpose $R_i^T : R^{n_i} \to R^n$ is a prolongation operator. Let $A_i = R_i A R_i^T$ denote the restriction of $A$ to $V_i$. Obviously, $A_i$ is nonsingular whenever $A$ is nonsingular. Moreover, if we choose the bases of $V_i$ appropriately, then the images of the bases of $V_i$ under the prolongation operator $R_i^T$ are linearly independent unit elements in $R^n$. In other words, the columns of $R_i^T$ consist of columns of the $n \times n$ identity matrix. Formally, such a matrix $R_i$ can be expressed as

$$R_i = [I_i \ 0]\pi_i \geq 0, \tag{2.1}$$

where $I_i$ is the $n_i \times n_i$ identity matrix and $\pi_i$ is some $n \times n$ permutation matrix. In this case, matrix $A_i$ is an $n_i \times n_i$ principal submatrix of $A$, which is also an M-matrix.

We describe steps of a general Schwarz method. Let $x^0$ be an initial approximation to the solution of (1.1). Generally, at step $k$, the additive Schwarz iteration scheme consists of the following substeps.

**Substep 1 (restriction):** Restrict the matrix $A$, the current residual $F - Ax^k$ and the vector $\phi - x^k$ as

$$A_i = R_i A R_i^T, \tag{2.2}$$

$$R_e^{k,i} = R_i(F - Ax^k), \tag{2.3}$$

$$\phi^{k,i} = R_i(\phi - x^k). \tag{2.4}$$

For each $i = 1, \ldots, m$, solve in parallel the local problem of finding $x^{k,i} \in R^{n_i}$ such that

$$x^{k,i} \geq \phi^{k,i}, \quad A_i x^{k,i} \geq R_e^{k,i}, \quad (x^{k,i} - \phi^{k,i})^T (A_i x^{k,i} - R_e^{k,i}) = 0. \tag{2.5}$$

**Substep 2 (prolongation):** Prolongate the approximations of errors by

$$x_e^{k,i} = R_i^T x^{k,i}, \qquad i = 1, 2, \ldots, m. \tag{2.6}$$

**Substep 3 (correction):** Correct $x^k$ to get

$$x^{k+1} = x^k + \sum_{i=1}^{m} \theta_i x_e^{k,i}, \tag{2.7}$$

where $\theta_1, \ldots, \theta_m$ are given positive weights.

**Remark 1** *For the linear equation (1.2), if $\theta_1 = \cdots = \theta_m = \theta$, then the above additive Schwarz iteration reduces to the damped additive Schwarz iteration scheme proposed in [4]*

$$x^{k+1} = T_\theta x^k, \tag{2.8}$$

*where $T_\theta$ is the matrix defined by*

$$T_\theta = I - \theta \sum_{i=1}^{m} R_i^T A_i^{-1} R_i A.$$

*Let $\bar{x}$ be the unique solution of (1.2). Then the iteration error $\epsilon^{k+1} \triangleq x^{k+1} - \bar{x}$ for (2.8) satisfies*

$$\epsilon^{k+1} = T_\theta \epsilon^k.$$

*However, for the LCP (1.1), iteration (2.7) usually induces a nonlinear operator. In other words, the corresponding iteration error $\epsilon^{k+1} = x^{k+1} - \bar{x}$ for (2.7) satisfies*

$$\epsilon^{k+1} = \tilde{T}_\theta(\epsilon^k),$$

*where $\tilde{T}_\theta : R^n \to R^n$ is a nonlinear operator.*

**Remark 2** *If $\sum_{i=1}^{m} \theta_i = 1$, then iteration (2.7) reduces to the overlapping Jacobian decomposition method proposed in [9, 10] or additive Schwarz algorithm proposed in [13].*

The following concepts will play an important role in the subsequent analysis.

**Definition 1** *([7]) Let $w \in R^n$ be a positive vector. For a vector $y \in R^n$, the weighted max-norm is defined by*

$$\|y\|_w = \max_{1 \le j \le n} |\frac{y_j}{w_j}|.$$

*For a matrix $A \in R^{n \times n}$, the weighted max-norm is defined by*

$$\|A\|_w = \sup_{\|y\|_w = 1} \{\|Ay\|_w \mid y \in R^n\}.$$

Obviously, if $w = (1, \ldots, 1)^T$, then the weighted max-norm reduces to the usual maximum norm.

**Definition 2** *([16, 20]) For a matrix $A \in R^{n \times n}$, we call $A = M - N$ a weak regular splitting of $A$ if $M^{-1} \ge 0$ and $M^{-1}N \ge 0$.*

The above definition of weak regular splitting has been widely used to analyze the convergence of various splitting algorithms [14, 15]. Later, we will use this concept to estimate the weighted max-norm bounds for the proposed Schwarz additive iteration scheme.

We conclude this section by giving some notations that will be used throughout the paper. Let $I, J \subset \{1, \ldots, n\}$ be index sets. For a matrix $A = (a_{ij}) \in R^{n \times n}$ and a vector $x = (x_i) \in R^n$, we let $A_{IJ}$ denote the submatrix of $A$ with elements $a_{ij}$ $(i \in I, j \in J)$ and $x_I$ denote the subvector of $x$ with elements $x_i$ $(i \in I)$. We denote by $|A|$ and $|x|$ the matrix $(|a_{ij}|)$ and the vector $(|x_i|)$, respectively. Similarly, the matrix inequalities $A \ge B$ and $A > B$, and the vector inequalities $x \ge y$ and $x > y$ are understood element-wise. Let $\pi \in R^{n \times n}$ be a permutation matrix. We denote $A_\pi = \pi A \pi^T$, $x_\pi = \pi x$, $F_\pi = \pi F$, and $\phi_\pi = \pi \phi$. For $i = 1, \ldots, m$, since

$$A_i = R_i A R_i^T = \begin{bmatrix} I_i & 0 \end{bmatrix} A_{\pi_i} \begin{bmatrix} I_i \\ 0 \end{bmatrix}$$

is the $n_i \times n_i$ submatrix of $A_{\pi_i}$, we can represent matrix $A_{\pi_i}$ in the form

$$A_{\pi_i} = \begin{bmatrix} A_i & G_i \\ H_i & A_{i_c} \end{bmatrix}. \tag{2.9}$$

Also it will be found convenient to represent vectors $x_{\pi_i}$, $F_{\pi_i}$ and $\phi_{\pi_i}$ as

$$x_{\pi_i} = \begin{bmatrix} u_i \\ u_{i_c} \end{bmatrix}, \quad F_{\pi_i} = \begin{bmatrix} f_i \\ f_{i_c} \end{bmatrix}, \quad \phi_{\pi_i} = \begin{bmatrix} \varphi_i \\ \varphi_{i_c} \end{bmatrix},$$

where $u_i = R_i x \in R^{n_i}$, $f_i = R_i F \in R^{n_i}$ and $\varphi_i = R_i \phi \in R^{n_i}$.

## 3. Preliminaries

In this section, we prove some useful properties for the iterative scheme (2.7). We suppose that $\{x_k\}$ is generated by (2.7).

From the special structure of permutation matrices, the following lemma is obvious.

**Lemma 3.1** *For any permutation matrix $\pi \in R^{n\times n}$, $A_\pi$ is also an $M$-matrix.*

It will be convenient to denote

$$E_i = R_i^T R_i, \quad E_\theta = \sum_{i=1}^{m} \theta_i E_i, \quad i = 1, 2, \ldots, m, \tag{3.1}$$

where for each $i$, $\theta_i$ is a positive constant and $R_i$ is defined by (2.1).

**Lemma 3.2** *Let $E_i$ and $E_\theta$ be defined by (3.1). Then we have for each $i = 1, 2, \ldots, m$, and any positive $\theta = (\theta_1, \ldots, \theta_m)^T$ that*

$$0 \le E_i \le I \quad \textit{and} \quad 0 \le E_\theta \le \sum_{i=1}^{m} \theta_i I. \tag{3.2}$$

*Moreover, matrix $E_\theta$ is a diagonal matrix with positive diagonal elements and hence is a nonsingular nonnegative diagonal matrix.*

**Proof** For each $i = 1, \ldots, m$, since $E_i = \pi_i^T \begin{bmatrix} I_i \\ 0 \end{bmatrix} \begin{bmatrix} I_i & 0 \end{bmatrix} \pi_i = \pi_i^T \begin{bmatrix} I_i & 0 \\ 0 & 0 \end{bmatrix} \pi_i$, $E_i$ is a nonnegative diagonal matrix whose diagonal elements are either one or zero. It is then easy to see that $E_i$ and $E_\theta = \sum\limits_{i=1}^{m} \theta_i E_i$ satisfy (3.2). If $E_\theta$ has a zero diagonal element, i.e., there exists an index $j$ such that the $j$-th diagonal element of each $E_i$ and hence the $j$-th row of each $R_i^T$ is zero, then the $j$-th element of vector $v = \sum\limits_{i=1}^{m} R_i^T w_i$ is always zero for any $w_i \in R^{n_i}$. This is a contradiction to (1.3). Therefore all diagonal elements of $E_\theta$ are positive and hence $E_\theta$ is nonsingular. $\Box$

**Lemma 3.3** *For $i = 1, 2, \ldots m$, we have*

$$A_i^{-1} \le R_i A^{-1} R_i^T. \tag{3.3}$$

**Proof** By (2.9), we have

$$A_{\pi_i} = \begin{bmatrix} A_i & G_i \\ H_i & A_{i_c} \end{bmatrix} \le \begin{bmatrix} A_i & 0 \\ 0 & A_{i_c} \end{bmatrix} \triangleq \tilde{A}_{\pi_i}, \tag{3.4}$$

where the inequality holds because $A_{\pi_i}$ is an M-matrix by Lemma 3.1. It is clear that $\tilde{A}_{\pi_i}$ is also an M-matrix. It then follows from (3.4) that $\tilde{A}_{\pi_i}^{-1} \leq A_{\pi_i}^{-1}$[16, Th.2.4.10]. Moreover, we have

$$A_i^{-1} = \begin{bmatrix} I_i & 0 \end{bmatrix} \tilde{A}_{\pi_i}^{-1} \begin{bmatrix} I_i \\ 0 \end{bmatrix} \leq \begin{bmatrix} I_i & 0 \end{bmatrix} A_{\pi_i}^{-1} \begin{bmatrix} I_i \\ 0 \end{bmatrix} = \begin{bmatrix} I_i & 0 \end{bmatrix} \pi_i A^{-1} \pi_i^T \begin{bmatrix} I_i \\ 0 \end{bmatrix} = R_i A^{-1} R_i^T.$$

The proof is complete. $\square$

The following lemma shows that if at some step $k$, $x^k$ coincides with the unique solution of problem (1.1), then $0 \in R^{n_i}$ is the unique solution of problem (2.5) for each $i = 1, 2, \ldots, m$.

**Lemma 3.4** *Let $\bar{x}$ be the unique solution of (1.1). If $x^k = \bar{x}$, then we have $x^{k,i} = 0$, $i = 1, \ldots, m$.*

**Proof** Since $\bar{x} - \phi \geq 0$, $A\bar{x} - F \geq 0$ and $(\bar{x} - \phi)^T (A\bar{x} - F) = 0$, it follows from the nonnegativity of $R_i$ that

$$\begin{cases} 0 - \phi^{k,i} = R_i(\bar{x} - \phi) \geq 0, \\ A_i 0 - R_e^{k,i} = R_i(A\bar{x} - F) \geq 0. \end{cases} \tag{3.5}$$

Multiplying these two inequalities and noting (3.1) and (3.2), we have

$$0 \leq (0 - \phi^{k,i})^T (A_i 0 - R_e^{k,i}) = (\bar{x} - \phi)^T R_i^T R_i (A\bar{x} - F) \leq (\bar{x} - \phi)^T (A\bar{x} - F) = 0,$$

and hence

$$(0 - \phi^{k,i})^T (A_i 0 - R_e^{k,i}) = 0. \tag{3.6}$$

It follows from (3.5) and (3.6) that $x^{k,i} = 0$ is a solution of (2.5), which is unique since $A_i$ is an M-matrix. The proof is complete. $\square$

**Lemma 3.5** *Let $x_{\pi_i}^k = \begin{bmatrix} u_i^k \\ u_{i_c}^k \end{bmatrix}$ with $u_i^k = R_i x^k \in R^{n_i}$, and $y^{k,i} \in R^{n_i}$ be given by $y^{k,i} = x^{k,i} + u_i^k$. Then $y^{k,i}$ is the solution of the following LCP on $R^{n_i}$:*

$$y \geq \varphi_i, \quad A_i y - F^{k,i} \geq 0, \quad (y - \varphi_i)^T (A_i y - F^{k,i}) = 0, \tag{3.7}$$

*where $\varphi_i = R_i \phi$, $F^{k,i} = f_i - G_i u_{i_c}^k$ and $f_i = R_i F$.*

**Proof** By the definition of $y^{k,i}$, we have

$$y^{k,i} - \varphi_i = x^{k,i} + u_i^k - \varphi_i = x^{k,i} - R_i(\phi - x^k) = x^{k,i} - \phi^{k,i}.$$

Since $\pi_i^T \pi_i = I$, it follows from (2.9) that

$$\begin{aligned} A_i y^{k,i} - F^{k,i} &= A_i x^{k,i} + A_i u_i^k + G_i u_{i_c}^k - f_i \\ &= A_i x^{k,i} + [A_i \ \ G_i] x_{\pi_i}^k - R_i F \end{aligned}$$

$$
\begin{aligned}
&= A_i x^{k,i} + [I_i \ \ 0] A_{\pi_i} x^k_{\pi_i} - R_i F \\
&= A_i x^{k,i} + [I_i \ \ 0] \pi_i A \pi_i^T \pi_i x^k - R_i F \\
&= A_i x^{k,i} + R_i (A x^k - F) \\
&= A_i x^{k,i} - R_e^{k,i}.
\end{aligned}
$$

Consequently, (3.7) follows from (2.5). $\square$

Lemmas 3.4 and 3.5 imply the following useful corollary.

**Corollary 3.1** *Let $\bar{x}$ be the solution of (1.1). Then $\bar{u}_i = R_i \bar{x}$ is the unique solution of the following LCP on $R^{n_i}$:*

$$
y \geq \varphi_i, \quad A_i y - F^{*,i} \geq 0, \quad (y - \varphi_i)^T (A_i y - F^{*,i}) = 0,
$$

*where $\varphi_i = R_i \phi$, $F^{*,i} = f_i - G_i \bar{u}_{i_c}$ and $f_i = R_i F$.*

## 4. Monotone Convergence

In this section, we prove the monotone convergence property of the iterative scheme (2.7). We first recall the concept of super-solution [6]. The super-solution set of problem (1.1) is the set

$$
S = \{ y \in R^n : y \geq \phi, \ Ay - F \geq 0 \}. \tag{4.1}
$$

This set is also called the feasible set of (1.1) in the LCP literature. It is well known that the solution $\bar{x}$ of problem (1.1) is a minimal element of super-solution set $S$ if $A$ is an M-matrix. In the following, we prove monotone convergence of $\{x^k\}$ after showing some lemmas.

**Lemma 4.1** *Let $x^{k,i}$ be the solution of (2.5). If $x^k \in S$, then inequality $x^{k,i} \leq 0$ holds for each $i = 1, \ldots, m$.*

**Proof** Since $Ax^k - F \geq 0$ and $x^k \geq \phi$, it follows from (2.1), (2.3) and (2.4) that $R_e^{k,i} \leq 0$ and $\phi^{k,i} \leq 0$. This implies that $0 \in R^{n_i}$ is a super-solution of problem (2.5). Since for each $i = 1, \ldots, m$, $A_i$ is also an M-matrix, it follows that $x^{k,i}$ is a minimal element of the super-solution set of problem (2.5) and hence we have $x^{k,i} \leq 0$. $\square$

**Lemma 4.2** *If $\sum\limits_{i=1}^{m} \theta_i \leq 1$ and $x^k \in S$, then we have $x^{k+1} \in S$.*

**Proof** Let $R_i$, $E_i$ and $E_\theta$ be defined by (2.1) and (3.1), respectively. By (2.4) and (2.5), we have $x^{k,i} \geq R_i(\phi - x^k)$. It then follows from Lemma 3.2 that

$$
x^{k+1} = x^k + \sum_{i=1}^{m} \theta_i R_i^T x^{k,i}
$$

$$
\begin{aligned}
&\geq x^k + \sum_{i=1}^{m} \theta_i R_i^T R_i(\phi - x^k) \\
&= x^k + \sum_{i=1}^{m} \theta_i E_i(\phi - x^k) \\
&= x^k + E_\theta(\phi - x^k) \\
&= \phi + (I - E_\theta)(x^k - \phi) \\
&\geq \phi,
\end{aligned} \tag{4.2}
$$

where the last inequality follows from (3.2) and the given assumption of this lemma. Since the equalities $\pi^T\pi = \pi\pi^T = I$ hold for any permutation matrix $\pi$, we get

$$
\begin{aligned}
Ax^{k+1} - F &= A(x^k + \sum_{i=1}^{m} \theta_i R_i^T x^{k,i}) - F \\
&= Ax^k - F + \sum_{i=1}^{m} \theta_i A \pi_i^T \begin{bmatrix} I_i \\ 0 \end{bmatrix} x^{k,i} \\
&= Ax^k - F + \sum_{i=1}^{m} \theta_i \pi_i^T A_{\pi_i} \begin{bmatrix} I_i \\ 0 \end{bmatrix} x^{k,i} \\
&= Ax^k - F + \sum_{i=1}^{m} \theta_i \pi_i^T \begin{bmatrix} A_i \\ H_i \end{bmatrix} x^{k,i},
\end{aligned} \tag{4.3}
$$

where the last equality follows from (2.9). By Lemma 3.1, for each $i = 1, \ldots, m$, $A_{\pi_i}$ is an M-matrix, which implies $H_i \leq 0$. It then follows from (4.3) and Lemma 4.1 that

$$
\begin{aligned}
Ax^{k+1} - F &\geq Ax^k - F + \sum_{i=1}^{m} \theta_i \pi_i^T \begin{bmatrix} A_i \\ 0 \end{bmatrix} x^{k,i} \\
&= Ax^k - F + \sum_{i=1}^{m} \theta_i \pi_i^T \begin{bmatrix} I_i \\ 0 \end{bmatrix} A_i x^{k,i} \\
&= Ax^k - F + \sum_{i=1}^{m} \theta_i R_i^T A_i x^{k,i} \\
&\geq Ax^k - F + \sum_{i=1}^{m} \theta_i R_i^T R_e^{k,i} \\
&= Ax^k - F + \sum_{i=1}^{m} \theta_i R_i^T R_i(F - Ax^k) \\
&= Ax^k - F + \sum_{i=1}^{m} \theta_i E_i(F - Ax^k) \\
&\geq (1 - \sum_{i=1}^{m} \theta_i)(Ax^k - F) \\
&\geq 0,
\end{aligned} \tag{4.4}
$$

where the second inequality follows from (2.5), and the third inequality follows from Lemma 3.2 and the condition $x^k \in S$. Inequality (4.2) together with inequality (4.4) implies $x^{k+1} \in S$ as desired. $\square$

The following theorem shows monotone convergence of the additive Schwarz iteration scheme.

**Theorem 4.1** *Let positive constants $\theta_1, \dots, \theta_m$ satisfy $\sum\limits_{i=1}^{m} \theta_i \leq 1$. If $x^0 \in S$, then $\{x^k\}$ converges to the solution $\bar{x}$ of (1.1). Moreover, we have for any $k \geq 0$*

$$x^k \in S \quad \text{and} \quad \bar{x} \leq x^{k+1} \leq x^k. \tag{4.5}$$

**Proof** Since $x^0 \in S$, it follows from Lemma 4.2 that $x^k \in S$ holds for any $k$. Moreover, it follows from (2.6), (2.7) and Lemma 4.1 that $x^{k+1} \leq x^k$ holds for any $k$. In particular, $\{x^k\}$ is convergent. Let $x^k \to x^*$. Clearly we have $x^* \leq x^k$ for all $k$. Since $x^{k,i} \leq 0$ by Lemma 4.1, we get $x_e^{k,i} \leq 0$ by (2.6). Taking limits on both sides of (2.7) yields $x_e^{k,i} \to 0$ as $k \to \infty$. Since $\mathrm{rank}(R_i^T) = n_i$, it follows from (2.6) that $x^{k,i} \to 0$ as $k \to \infty$. Therefore, from (2.5), we deduce

$$-R_i(\phi - x^*) \geq 0, \quad -R_i(F - Ax^*) \geq 0$$

and

$$(-R_i(\phi - x^*))^T(-R_i(F - Ax^*)) = 0.$$

It then follows that for each $i = 1, 2, \dots, m$

$$R_i^T R_i(x^* - \phi) \geq 0, \quad R_i^T R_i(Ax^* - F) \geq 0 \tag{4.6}$$

and

$$(x^* - \phi)^T R_i^T R_i(Ax^* - F) = 0. \tag{4.7}$$

Let $E = \sum_{i=1}^m E_i = \sum_{i=1}^m R_i^T R_i$. Summing inequalities (4.6) and equalities (4.7) over $i = 1, 2, \dots m$, respectively, we get

$$E(x^* - \phi) \geq 0, \quad E(Ax^* - F) \geq 0 \tag{4.8}$$

and

$$(x^* - \phi)^T E(Ax^* - F) = 0. \tag{4.9}$$

By Lemma 3.2, $E$ is a diagonal matrix with positive diagonals. Therefore, (4.8) and (4.9) are equivalent to

$$x^* - \phi \geq 0, \quad Ax^* - F \geq 0, \quad (x^* - \phi)^T(Ax^* - F) = 0.$$

This shows that $x^*$ is the solution of (1.1). $\square$

## 5. Weighted Max-Norm Bounds

In this section, we estimate the weighted max-norm bounds for iteration errors of the scheme (2.7). First, we cite a lemma from [4].

**Lemma 5.1** *Let $P$ be a matrix, $w$ be a positive vector and $\gamma$ be a positive scalar such that*

$$|P|w \leq \gamma w. \tag{5.1}$$

*Then we have $\|P\|_w \leq \gamma$. In particular, we have $\|Px\|_w \leq \gamma\|x\|_w$ for all $x$. Moreover, if strict inequality holds in (5.1), then we have $\|P\|_w < \gamma$.*

We prove some useful lemmas.

**Lemma 5.2** *Let $A = M_i - N_i$, $i = 1, \dots m$, be weak regular splittings of $A$, and $E_i$, $i = 1, \dots, m$, be diagonal matrices given by (3.1). Then the following statements hold true for any positive constants $\theta_1, \dots, \theta_m$.*

*(a) Matrix $P_\theta \stackrel{\triangle}{=} E_\theta - \sum_{i=1}^{m} \theta_i E_i M_i^{-1} A$ is nonnegative.*

*(b) There exists a positive vector $w \in R^n$ such that $\|E_\theta^{-1} P_\theta\|_w < 1$. In particular, we have $\rho(E_\theta^{-1} P_\theta) < 1$, where $\rho(B)$ denotes the spectral radius of matrix $B$.*

*(c) Matrix $\sum_{i=1}^{m} \theta_i E_i M_i^{-1}$ is nonnegative and nonsingular.*

**Proof** Notice that $P_\theta = \sum_{i=1}^{m} \theta_i E_i (I - M_i^{-1} A) = \sum_{i=1}^{m} \theta_i E_i M_i^{-1} N_i$. Statement (a) then follows form the definition of weak regular splitting.

Let $e = (1, \dots 1)^T$ and $w = A^{-1}e$. Since $A^{-1}$ exists and is nonnegative, it is clear that $w > 0$ and $M_i^{-1} e > 0, i = 1, \dots m$. Therefore, we get

$$P_\theta w = E_\theta w - \sum_{i=1}^{m} \theta_i E_i M_i^{-1} e < E_\theta w,$$

which implies

$$E_\theta^{-1} P_\theta w < w.$$

By Lemma 5.1, we have $\|E_\theta^{-1} P_\theta\|_w < 1$. Let $\lambda$ be the eigenvalue of $E_\theta^{-1} P_\theta$ such that $|\lambda| = \rho(E_\theta^{-1} P_\theta)$. Then there exists a nonzero vector $y$ such that $E_\theta^{-1} P_\theta y = \lambda y$ and hence $\rho(E_\theta^{-1} P_\theta) = \frac{\|E_\theta^{-1} P_\theta y\|_w}{\|y\|_w} \leq \|E_\theta^{-1} P_\theta\|_w < 1$. Therefore, statement (b) is true.

To prove (c), it suffices to verify the nonsingularity of matrix $\sum_{i=1}^{m} \theta_i E_i M_i^{-1}$, because the nonnegativity is obvious. It follows from statement (b) that

$$\rho(I - (E_\theta^{-1} \sum_{i=1}^{m} \theta_i E_i M_i^{-1} A)) = \rho(E_\theta^{-1} P_\theta) < 1.$$

This implies that matrix $E_\theta^{-1}\sum_{i=1}^{m}\theta_i E_i M_i^{-1}A$ is nonsingular and hence $\sum_{i=1}^{m}\theta_i E_i M_i^{-1}$ is nonsingular. The proof is complete. $\Box$

**Lemma 5.3** *For any positive constants $\theta_1,\ldots,\theta_m$, matrix $B_\theta \triangleq \sum_{i=1}^{m}\theta_i R_i^T A_i^{-1}R_i = \sum_{i=1}^{m}\theta_i E_i M_i^{-1}$ is nonnegative and nonsingular.*

**Proof** Let $E_i$ be defined by (3.1). Then $E_i$ is diagonal and $\sum_{i=1}^{m}E_i$ is nonsingular by Lemma 3.2. Let

$$M_i = \pi_i^T \begin{bmatrix} A_i & 0 \\ 0 & A_{i_c} \end{bmatrix} \pi_i, \qquad i = 1,\ldots,m. \tag{5.2}$$

Since $A_{\pi_i}$ is an M-matrix, it follows from (5.2) that

$$M_i^{-1} = \pi_i^T \begin{bmatrix} A_i^{-1} & 0 \\ 0 & A_{i_c}^{-1} \end{bmatrix} \pi_i \geq 0. \tag{5.3}$$

Let $N_i = M_i - A$. Then we have

$$N_i = M_i - \pi_i^T A_{\pi_i}\pi_i \geq \pi_i^T \begin{bmatrix} A_i & 0 \\ 0 & A_{i_c} \end{bmatrix} \pi_i - \pi_i^T \begin{bmatrix} A_i & 0 \\ 0 & A_{i_c} \end{bmatrix} \pi_i = 0, \tag{5.4}$$

where the inequality holds because $A_{\pi_i}$ is an M-matrix. Inequalities (5.3) and (5.4) imply that for each $i = 1,\ldots,m$, $A = M_i - N_i$ is a weak regular splitting of $A$. Moreover, we have

$$\begin{aligned} E_i M_i^{-1} &= R_i^T R_i \pi_i^T \begin{bmatrix} A_i^{-1} & 0 \\ 0 & A_{i_c}^{-1} \end{bmatrix} \pi_i \\ &= R_i^T [I_i \;\; 0] \begin{bmatrix} A_i^{-1} & 0 \\ 0 & A_{i_c}^{-1} \end{bmatrix} \pi_i \\ &= R_i^T [A_i^{-1} \;\; 0]\pi_i \\ &= R_i^T A_i^{-1}[I_i \;\; 0]\pi_i \\ &= R_i^T A_i^{-1} R_i. \end{aligned} \tag{5.5}$$

By the use of Lemma 5.2 (c), we conclude that matrix $B_\theta = \sum_{i=1}^{m}\theta_i R_i^T A_i^{-1}R_i = \sum_{i=1}^{m}\theta_i E_i M_i^{-1}$ is nonnegative and nonsingular. $\Box$

**Lemma 5.4** *Let $\bar{x}$ be the unique solution of (1.1) and let $\bar{x}_{\pi_i} = \begin{bmatrix} \bar{u}_i \\ \bar{u}_{i_c} \end{bmatrix}, i = 1,\ldots,m$. Denote $y^{k,i} = x^{k,i} + \bar{u}_i$ and $y^{*,i} = \bar{u}_i$. Then we have*

$$A_i|y^{k,i} - y^{*,i}| \leq -G_i|u_{i_c}^k - \bar{u}_{i_c}|. \tag{5.6}$$

**Proof** We show that (5.6) holds element-wise. We consider three cases.

Case I: $(y^{k,i})_j > (y^{*,i})_j$. Since $\bar{x} \geq \phi$, it is obvious that $(y^{k,i})_j > (\varphi_i)_j$, where $y^{*,i} = \bar{u}_i = R_i\bar{x}$ and $\varphi_i = R_i\phi$. By Lemma 3.5 and Corollary 3.1, we have

$$(A_i y^{k,i} - f_i + G_i u^k_{i_c})_j = 0$$

and

$$(A_i y^{*,i} - f_i + G_i \bar{u}_{i_c})_j \geq 0.$$

It then follows that

$$(A_i(y^{k,i} - y^{*,i}))_j \leq -(G_i(u^k_{i_c} - \bar{u}_{i_c}))_j = (|G_i|(u^k_{i_c} - \bar{u}_{i_c}))_j,$$

where the equality follows from $G_i \leq 0$. Since $(y^{k,i} - y^{*,i})_j > 0$ and $A_i$ is an M-matrix, the last inequality together with $G_i \leq 0$ yields

$$(A_i|y^{k,i} - y^{*,i}|)_j \leq (|G_i|\,|u^k_{i_c} - \bar{u}_{i_c}|)_j = -(G_i|u^k_{i_c} - \bar{u}_{i_c}|)_j. \tag{5.7}$$

Case II: $(y^{k,i})_j < (y^{*,i})_j$. This implies $(y^{*,i})_j > (\varphi_i)_j$. It follows from Lemma 3.5 and Corollary 3.1 again that

$$(A_i y^{k,i} - f_i + G_i u^k_{i_c})_j \geq 0$$

and

$$(A_i y^{*,i} - f_i + G_i \bar{u}_{i_c})_j = 0.$$

In a way similar to Case I, we also get (5.7).

Case III: $(y^{k,i})_j = (y^{*,i})_j$. In this case, since $A_i$ is a matrix with nonpositive off-diagonal elements, the inequality (5.7) follows from the fact that the left-hand side is nonpositive while the right-hand side is nonnegative. □

The following theorem gives a weighted norm estimate of iteration errors.

**Theorem 5.1** *Let $\theta_1, \ldots, \theta_m$ be positive constants satisfying $\sum_{i=1}^m \theta_i \leq 1$, and $\epsilon^k \triangleq x^k - \bar{x}$ be the iteration error of (2.7), where $\bar{x}$ is the solution of (1.1). Then we have*

$$0 \leq |\epsilon^{k+1}| \leq T_\theta|\epsilon^k|,$$

*where $T_\theta = I - \sum_{i=1}^m \theta_i R_i^T A_i^{-1} R_i A$ is a nonnegative matrix. Moreover, there exists a positive vector $w$ and a scalar $\gamma \in (0,1)$ such that*

$$\|T_\theta\|_w \leq \gamma \tag{5.8}$$

*and hence*

$$\|\epsilon^{k+1}\|_w \leq \|T_\theta \epsilon^k\|_w \leq \gamma\|\epsilon^k\|_w. \tag{5.9}$$

**Proof** Let $E_i$ and $M_i$ be defined by (3.1) and (5.2), respectively. Then we have

$$T_\theta = I - \sum_{i=1}^{m} \theta_i E_i M_i^{-1} A \geq E_\theta - \sum_{i=1}^{m} \theta_i E_i M_i^{-1} A = P_\theta \geq 0,$$

where the first equality follows from (5.5), the first inequality follows from Lemma 3.2 and $\sum_{i=1}^m \theta_i \leq 1$, and the last inequality follows from Lemma 5.2 (a). Let $e = (1, \ldots, 1)^T \in R^n$ and $w = A^{-1}e$. Then it is clear that $w > 0$. Moreover, we get from Lemma 5.3

$$T_\theta w = w - \sum_{i=1}^{m} \theta_i E_i M_i^{-1} A w = w - B_\theta e < w.$$

Therefore there exists a constant $\gamma \in (0, 1)$ such that $T_\theta w \leq \gamma w$. Inequality (5.8) then follows from Lemma 5.1. On the other hand, we deduce from (2.6) that

$$\begin{aligned}
0 \;\leq\; & |\epsilon^{k+1}| = |\epsilon^k + \sum_{i=1}^{m} \theta_i x_e^{k,i}| \\
= \; & |(1 - \sum_{i=1}^{m} \theta_i)\epsilon^k + \sum_{i=1}^{m} \theta_i (R_i^T x^{k,i} + \epsilon^k)| \\
= \; & \left| (1 - \sum_{i=1}^{m} \theta_i)\epsilon^k + \sum_{i=1}^{m} \theta_i \pi_i^T \left( \begin{bmatrix} x^{k,i} \\ 0 \end{bmatrix} + \begin{bmatrix} u_i^k - \bar{u}_i \\ u_{i_c}^k - \bar{u}_{i_c} \end{bmatrix} \right) \right| \\
= \; & \left| (1 - \sum_{i=1}^{m} \theta_i)\epsilon^k + \sum_{i=1}^{m} \theta_i \pi_i^T \begin{bmatrix} y^{k,i} - y^{*,i} \\ u_{i_c}^k - \bar{u}_{i_c} \end{bmatrix} \right| \\
\leq \; & |(1 - \sum_{i=1}^{m} \theta_i)\epsilon^k| + \sum_{i=1}^{m} \theta_i \pi_i^T \begin{bmatrix} |y^{k,i} - y^{*,i}| \\ |u_{i_c}^k - \bar{u}_{i_c}| \end{bmatrix} \\
\leq \; & |1 - \sum_{i=1}^{m} \theta_i||\epsilon^k| + \sum_{i=1}^{m} \theta_i \pi_i^T \begin{bmatrix} -A_i^{-1} G_i |u_{i_c}^k - \bar{u}_{i_c}| \\ |u_{i_c}^k - \bar{u}_{i_c}| \end{bmatrix},
\end{aligned}$$

where $y^{k,i} = x^{k,i} + u_i^k$, $y^{*,i} = \bar{u}_i$, the second equality follows from (2.1), and the last inequality follows from (5.6) and $A_i^{-1} \geq 0$. Since $\sum\limits_{i=1}^{m} \theta_i \leq 1$, we deduce from the above formula that

$$\begin{aligned}
0 \;\leq\; & |\epsilon^{k+1}| \\
\leq \; & \Big(1 - \sum_{i=1}^{m} \theta_i\Big)|\epsilon^k| + \sum_{i=1}^{m} \theta_i \pi_i^T \begin{bmatrix} 0 & -A_i^{-1} G_i \\ 0 & I_{i_c} \end{bmatrix} |\epsilon_{\pi_i}^k| \\
= \; & |\epsilon^k| - \sum_{i=1}^{m} \theta_i \pi_i^T |\epsilon_{\pi_i}^k| + \sum_{i=1}^{m} \theta_i \pi_i^T \begin{bmatrix} 0 & -A_i^{-1} G_i \\ 0 & I_{i_c} \end{bmatrix} |\epsilon_{\pi_i}^k| \\
= \; & |\epsilon^k| + \sum_{i=1}^{m} \theta_i \pi_i^T \begin{bmatrix} -I_i & -A_i^{-1} G_i \\ 0 & 0 \end{bmatrix} |\epsilon_{\pi_i}^k|
\end{aligned}$$

$$
\begin{aligned}
&= |\epsilon^k| - \sum_{i=1}^{m} \theta_i R_i^T A_i^{-1} [A_i \ \ G_i] |\epsilon_{\pi_i}^k| \\
&= |\epsilon^k| - \sum_{i=1}^{m} \theta_i R_i^T A_i^{-1} [I_i \ \ 0] A_{\pi_i} |\epsilon_{\pi_i}^k| \\
&= |\epsilon^k| - \sum_{i=1}^{m} \theta_i R_i^T A_i^{-1} [I_i \ \ 0] \pi_i A \pi_i^T \pi_i |\epsilon^k| \\
&= |\epsilon^k| - \sum_{i=1}^{m} \theta_i R_i^T A_i^{-1} R_i A |\epsilon^k| \\
&= T_\theta |\epsilon^k|,
\end{aligned}
$$

where the third and the sixth equalities follow from (2.1), and the fourth equality follows from (2.9) and (3.3). Therefore, we get

$$
\|\epsilon^{k+1}\|_w = \||\epsilon^{k+1}|\|_w \leq \|T_\theta |\epsilon^k|\|_w \leq \gamma \||\epsilon^k|\|_w = \gamma \|\epsilon^k\|_w,
$$

The proof is complete. $\square$

Theorem 5.1 provides a weighted norm estimate of iteration errors. Particularly, since $\gamma \in (0,1)$, we have the following global convergence result.

**Corollary 5.1** *Let $\theta_1, \ldots, \theta_m$ be positive constants satisfying $\sum_{i=1}^{m} \theta_i \leq 1$. Then the sequence $\{x^k\}$ generated by the additive Schwarz iteration scheme (2.7) converges to the solution of (1.1) for any initial point $x^0$.*

If positive vector $(\theta_1, \ldots, \theta_m)^T$ satisfies $\sum_{i=1}^{m} \theta_i = 1$, i.e., the case of overlapping Jacorbi decomposition method (see Remark 2), then by Theorem 5.1 and $\rho(T_\theta) \leq \|T_\theta\|_w$, we have $|\epsilon^{k+1}| \leq T_\theta |\epsilon^k|$ with $\rho(T_\theta) < 1$. This result has been obtained by Kuznetsov, Neittaanmäki, and Tarvainen [9, 10] for some special initial point.

## 6. Conclusion

We have developed an additive Schwarz iteration scheme for solving LCP. The proposed scheme contains some existing algorithms as special cases. We have proved monotone convergence of the proposed scheme. Theorem 5.1 shows that the weighted norm estimate for iteration errors does not rely on the choice of initial point, which particularly implies global convergence of the iteration scheme. Moreover, it is not difficult to see from Theorem 5.1 that convergence results of additive Schwarz method that hold for linear equations remain true for LCP.

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