線形計画

$\boxed{3}$

次の線形計画問題Pを考える．

$$\begin{array}{lll} \mathrm{P}: & \text{Minimize} & \boldsymbol{c}^\top \boldsymbol{x} \\ & \text{subject to} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & & \boldsymbol{x} \geqq \boldsymbol{0} \end{array}$$

ただし，$\boldsymbol{A}$ は $m \times n$ 定数行列，$\boldsymbol{b}$ は m 次元定数ベクトル，$\boldsymbol{c}$ は n 次元定数ベクトル，$\boldsymbol{x}$ は n 次元変数ベクトルであり，$^\top$ は転置記号を表す．さらに，問題(P)に関連して，非負パラメータ μ を含む次の条件 $\mathrm{Q}(\mu)$ を考える．

$$\mathrm{Q}(\mu): \begin{cases} \boldsymbol{A}^\top \boldsymbol{y} + \boldsymbol{z} = \boldsymbol{c} \\ \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ x_i z_i = \mu \ (i = 1, \ldots, n) \\ \boldsymbol{x} \geqq 0, \ \boldsymbol{z} \geqq 0 \end{cases}$$

ただし，$\boldsymbol{x} = (x_1, \ldots, x_n)^\top \in \mathbb{R}^n$, $\boldsymbol{y} = (y_1, \ldots, y_m)^\top \in \mathbb{R}^m$, $\boldsymbol{z} = (z_1, \ldots, z_n)^\top \in \mathbb{R}^n$ である．各 μ に対して，条件 $\mathrm{Q}(\mu)$ を満たすベクトル $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ は唯一存在すると仮定し，それらを $\boldsymbol{x}(\mu), \boldsymbol{y}(\mu), \boldsymbol{z}(\mu)$ と表す．

以下の問いに答えよ．

(i) 問題Pの双対問題をかけ．

(ii) 関数 $h : [0, \infty) \to \mathbb{R}$ を $h(\mu) = \boldsymbol{c}^\top \boldsymbol{x}(\mu) - \boldsymbol{b}^\top \boldsymbol{y}(\mu)$ と定義する．関数 h は $[0, \infty)$ 上で線形関数となることを示せ．

(iii) $\boldsymbol{x}(0)$ は問題Pの最適解となることを示せ．

(iv) $n = 2, m = 1$ とし，

$$\boldsymbol{A} = (1 \ \ 1), \quad \boldsymbol{b} = 1, \quad \boldsymbol{c} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

とする．このとき，任意の非負パラメータ μ に対して，条件 $\mathrm{Q}(\mu)$ を満たすベクトル $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ は唯一存在する．$\boldsymbol{x}(\mu)$ を求めよ．さらに，問(i)で与えた双対問題の最適解を求めよ．

An English Translation:

Linear Programming

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Consider the following linear programming:

$$\begin{array}{lll} \mathrm{P}: & \text{Minimize} & \boldsymbol{c}^\top \boldsymbol{x} \\ & \text{subject to} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & & \boldsymbol{x} \geqq \boldsymbol{0}, \end{array}$$

where $\boldsymbol{A}$ is an $m \times n$ matrix of constants, $\boldsymbol{b}$ is an m-dimensional vector of constants, $\boldsymbol{c}$ is an n-dimensional vector of constants, $\boldsymbol{x}$ is an n-dimensional vector of variables, and the superscript $^\top$ denotes transposition. Moreover, consider the following conditions $\mathrm{Q}(\mu)$ with a nonnegative parameter μ:

$$\mathrm{Q}(\mu): \begin{cases} \boldsymbol{A}^\top \boldsymbol{y} + \boldsymbol{z} = \boldsymbol{c} \\ \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ x_i z_i = \mu \ (i = 1, \ldots, n) \\ \boldsymbol{x} \geqq 0, \ \boldsymbol{z} \geqq 0, \end{cases}$$

where $\boldsymbol{x} = (x_1, \ldots, x_n)^\top \in \mathbb{R}^n$, $\boldsymbol{y} = (y_1, \ldots, y_m)^\top \in \mathbb{R}^m$ and $\boldsymbol{z} = (z_1, \ldots, z_n)^\top \in \mathbb{R}^n$. Suppose that, for each μ, there exist unique vectors $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ that satisfy conditions $\mathrm{Q}(\mu)$. Let $\boldsymbol{x}(\mu)$, $\boldsymbol{y}(\mu)$ and $\boldsymbol{z}(\mu)$ be these unique vectors for each μ.

Answer the following questions.

(i) Write a dual problem of problem P.

(ii) Let a function $h : [0, \infty) \to \mathbb{R}$ be defined by $h(\mu) = \boldsymbol{c}^\top \boldsymbol{x}(\mu) - \boldsymbol{b}^\top \boldsymbol{y}(\mu)$. Then show that the function h is linear on $[0, \infty)$.

(iii) Show that $\boldsymbol{x}(0)$ is an optimal solution to problem P.

(iv) Let $n = 2, m = 1$, $\boldsymbol{A} = (1 \ \ 1)$, $\boldsymbol{b} = 1$ and

$$\boldsymbol{c} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Then, for each nonnegative μ, there exist unique vectors $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ that satisfy conditions $\mathrm{Q}(\mu)$. Obtain $\boldsymbol{x}(\mu)$. Moreover, obtain an optimal solution of the dual problem given in question (i).