物理統計学

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1次元ランダムウォークを考える．ランダムウォーカーは1直線上に等間隔に並んだ点 $\ldots, x_{-1}, x_0, x_1, \ldots$ のどこかにいるものとする．ここで，$x_k = ak \ (k = \ldots, -1, 0, 1, \ldots)$ で a は正の定数．ウォーカーは初期には x_0 におり，時間 τ ごとに右か左の最近接点に，それぞれ確率 p，$q = 1 - p$ でジャンプすることを繰りかえす．ここで，$0 < p < 1$ である．すなわち，時刻 $t = n\tau \ (n = 1, 2, \ldots)$ にウォーカーはジャンプする．$P_n(k)$ を n 回のジャンプの後ウォーカーが x_k にいる確率とする．以下の問いに答えよ．

(i) $P_n(k)$ を求めよ．

(ii) $P_n(k)$ の n に関する漸化式を求めよ．

(iii) $f(x,t) := \dfrac{1}{2a} P_{\frac{t}{\tau}}\left(\dfrac{x}{a}\right)$ とし，$f(x,t)$ は t に関して連続的に微分可能で x に関して連続的に2階微分可能であると仮定する．$f(x,t)$ に対する偏微分方程式を，問 (ii) で求めた $P_n(k)$ の漸化式から，$\dfrac{a^2}{2\tau} = D$，$\dfrac{(p-q)a}{\tau} = v$ という条件のもとで，$a \to 0$，$\tau \to 0$ の極限をとって導出せよ．ここで，v は定数で，D は正の定数である．

(iv) ウォーカーの時刻 t での位置の平均 $X(t) := \displaystyle\int_{-\infty}^{\infty} \mathrm{d}x \ x f(x,t)$ と位置の分散 $\sigma^2(t) := -(X(t))^2 + \displaystyle\int_{-\infty}^{\infty} \mathrm{d}x \ x^2 f(x,t)$ を計算せよ．ここで，$\displaystyle\lim_{x\to\pm\infty} x^2 f(x,t) = \lim_{x\to\pm\infty} x^2 \frac{\partial f(x,t)}{\partial x} = 0$，$X(0) = \sigma^2(0) = 0$ および $\displaystyle\int_{-\infty}^{\infty} \mathrm{d}x \ f(x,t) = 1$ を用いてよい．

An English Translation:

Physical Statistics

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Let us consider the one-dimensional random walk. A random walker can be at regularly spaced positions $\ldots, x_{-1}, x_0, x_1, \ldots$ along a line, where $x_k = ak$ with $k = \ldots, -1, 0, 1, \ldots$ and a is a positive constant. The walker is at x_0 initially, and sequentially jumps to the nearest right or left position with probability p or $q = 1-p$, respectively, where $0 < p < 1$. Every time interval between the successive jumps equals τ, *i.e.* the walker jumps at time $t = n\tau$ $(n = 1, 2, \ldots)$. $P_n(k)$ denotes the probability that the walker is at x_k after n jumps. Answer the following questions.

(i) Obtain $P_n(k)$.

(ii) Obtain the recurrence formula of $P_n(k)$ for n.

(iii) Let $f(x,t) := \dfrac{1}{2a} P_{\frac{t}{\tau}}\left(\dfrac{x}{a}\right)$. $f(x,t)$ is assumed to be continuously differentiable with respect to t and twice continuously differentiable with respect to x. Derive the partial differential equation for $f(x,t)$ from the recurrence formula of $P_n(k)$ obtained in the question (ii) under the limit, $a \to 0$ and $\tau \to 0$ with $\dfrac{a^2}{2\tau} = D$ and $\dfrac{(p-q)a}{\tau} = v$, where v is a constant and D is a positive constant.

(iv) Calculate the average position $X(t)$ of the walker at time t, $X(t) := \int_{-\infty}^{\infty} dx\, x f(x,t)$ and the variance $\sigma^2(t)$ of the position at t, $\sigma^2(t) := -(X(t))^2 + \int_{-\infty}^{\infty} dx\, x^2 f(x,t)$ with the use of $\lim_{x\to\pm\infty} x^2 f(x,t) = \lim_{x\to\pm\infty} x^2 \dfrac{\partial f(x,t)}{\partial x} = 0$, $X(0) = \sigma^2(0) = 0$ and $\int_{-\infty}^{\infty} dx\, f(x,t) = 1$.