線形計画

$\boxed{3}$

関数 $f : \mathbb{R}^n \to \mathbb{R}$ を連続的微分可能な凸関数とする．さらに，$\nabla f(\boldsymbol{x})$ を次式で定義される関数 f の $\boldsymbol{x} \in \mathbb{R}^n$ における勾配とする．

$$\nabla f(\boldsymbol{x}) = \left(\frac{\partial f(\boldsymbol{x})}{\partial x_1}, \ldots, \frac{\partial f(\boldsymbol{x})}{\partial x_n} \right)^\top$$

ただし，$^\top$ は転置記号を表す．

次の線形計画問題Pを考える．

$$\begin{array}{lll} \mathrm{P}: & \text{Minimize} & \nabla f(\bar{\boldsymbol{x}})^\top \boldsymbol{x} \\ & \text{subject to} & \boldsymbol{A}(\bar{\boldsymbol{x}} + \boldsymbol{x}) \leqq \boldsymbol{b} \end{array}$$

ただし，$\boldsymbol{A}$ は $m \times n$ 定数行列，$\boldsymbol{b}$ は m 次元定数ベクトル，$\bar{\boldsymbol{x}}$ は n 次元定数ベクトル，$\boldsymbol{x}$ は n 次元変数ベクトルである．問題Pは最適解を持つとする．

以下の問いに答えよ．

(i) $\nabla f(\bar{\boldsymbol{x}})^\top \boldsymbol{d} \geqq 0$ である任意の $\boldsymbol{d} \in \mathbb{R}^n$ に対して $f(\bar{\boldsymbol{x}} + \boldsymbol{d}) \geqq f(\bar{\boldsymbol{x}})$ となることを示せ．

(ii) 問題Pの双対問題を書け．

(iii) $\boldsymbol{A}\bar{\boldsymbol{x}} \geqq \boldsymbol{b}$ とする．このとき，$\boldsymbol{A}\boldsymbol{z} \leqq \boldsymbol{b}$ である任意の $\boldsymbol{z} \in \mathbb{R}^n$ に対して，$f(\boldsymbol{z}) \geqq f(\bar{\boldsymbol{x}})$ であることを示せ．

An English Translation:

Linear Programming

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Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable convex function. Moreover, let $\nabla f(\boldsymbol{x})$ be the gradient of f at $\boldsymbol{x} \in \mathbb{R}^n$, which is defined by

$$\nabla f(\boldsymbol{x}) = \left(\frac{\partial f(\boldsymbol{x})}{\partial x_1}, \ldots, \frac{\partial f(\boldsymbol{x})}{\partial x_n} \right)^\top,$$

where the superscript $^\top$ denotes transposition of a vector.

Consider the following linear programming problem P.

$$\begin{array}{lll} \mathrm{P}: & \text{Minimize} & \nabla f(\bar{\boldsymbol{x}})^\top \boldsymbol{x} \\ & \text{subject to} & \boldsymbol{A}(\bar{\boldsymbol{x}} + \boldsymbol{x}) \leqq \boldsymbol{b}, \end{array}$$

where $\boldsymbol{A}$ is an $m \times n$ constant matrix, $\boldsymbol{b}$ is an m-dimensional constant vector, $\bar{\boldsymbol{x}}$ is an n-dimensional constant vector, and $\boldsymbol{x}$ is an n-dimensional vector of variables. Suppose that P has an optimal solution.

Answer the following questions.

(i) Show that $f(\bar{\boldsymbol{x}} + \boldsymbol{d}) \geqq f(\bar{\boldsymbol{x}})$ for any $\boldsymbol{d} \in \mathbb{R}^n$ such that $\nabla f(\bar{\boldsymbol{x}})^\top \boldsymbol{d} \geqq 0$.

(ii) Write out the dual problem of problem P.

(iii) Suppose that $\boldsymbol{A}\bar{\boldsymbol{x}} \geqq \boldsymbol{b}$. Then show that $f(\boldsymbol{z}) \geqq f(\bar{\boldsymbol{x}})$ for any $\boldsymbol{z} \in \mathbb{R}^n$ such that $\boldsymbol{A}\boldsymbol{z} \leqq \boldsymbol{b}$.