オペレーションズ・リサーチ

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関数 $h:\mathbb{R}^n \to \mathbb{R}$ を凸関数とする．さらに，関数 $g:\mathbb{R}\to\mathbb{R}$ と $f:\mathbb{R}^n\to\mathbb{R}$ を以下のように定義する．

$$g(t) = 2^t,\ \ f(\boldsymbol{x}) = g(h(\boldsymbol{x}))$$

ベクトル $\boldsymbol{b}^i \in \mathbb{R}^n\ (i=1,\ldots,m)$ が与えられたとき，集合 $\Delta \subsetneqq \mathbb{R}^n$，$\Gamma \subsetneqq \mathbb{R}^m$，$\Omega \subsetneqq \mathbb{R}^n$ を以下のように定義する．

$$\begin{aligned}
\Delta &= \left\{\boldsymbol{b}^1, \boldsymbol{b}^2, \ldots, \boldsymbol{b}^m\right\} \\
\Gamma &= \left\{\boldsymbol{\alpha} \in \mathbb{R}^m \,\middle|\, \sum_{i=1}^{m} \alpha_i = 1,\ \alpha_i \geqq 0\ (i=1,\ldots,m)\right\} \\
\Omega &= \left\{\boldsymbol{x} \in \mathbb{R}^n \,\middle|\, \boldsymbol{x} = \sum_{i=1}^{m} \alpha_i \boldsymbol{b}^i,\ \boldsymbol{\alpha} \in \Gamma\right\}
\end{aligned}$$

次の非線形計画問題 (P) を考える．

$$\begin{aligned}
\text{(P)}\quad &\text{Maximize} && f(\boldsymbol{x}) \\
&\text{subject to} && \boldsymbol{x} \in \Omega
\end{aligned}$$

以下の問いに答えよ．

(i) 任意の $\boldsymbol{\alpha} \in \Gamma$ に対して，次の不等式が成り立つことを示せ．

$$h\left(\sum_{i=1}^{m} \alpha_i \boldsymbol{b}^i\right) \leqq \sum_{i=1}^{m} \alpha_i h(\boldsymbol{b}^i)$$

(ii) 関数 g と f が凸関数であることを示せ．

(iii) 次の線形計画問題のカルーシュ・キューン・タッカー (Karush-Kuhn-Tucker) 条件を書け．

$$\begin{aligned}
&\text{Maximize} && \sum_{i=1}^{m} f(\boldsymbol{b}^i)\alpha_i \\
&\text{subject to} && \sum_{i=1}^{m} \alpha_i = 1 \\
& && \alpha_i \geqq 0\ \ (i=1,\ldots,m)
\end{aligned}$$

ただし，決定変数は $\alpha_i\ (i=1,\ldots,m)$ である．

(iv) 問題 (P) の最適解の集合を X^* とする．このとき，$X^* \cap \Delta \neq \emptyset$ となることを示せ．

An English Translation:

Operations Research

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Let $h : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Moreover, let $g : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R}^n \to \mathbb{R}$ be defined as $g(t) = 2^t$ and $f(\boldsymbol{x}) = g(h(\boldsymbol{x}))$, respectively.

For given vectors $\boldsymbol{b}^i \in \mathbb{R}^n$ $(i = 1, \ldots, m)$, let sets $\Delta \subseteqq \mathbb{R}^n$, $\Gamma \subseteqq \mathbb{R}^m$, and $\Omega \subseteqq \mathbb{R}^n$ be defined as

$$\begin{aligned}
\Delta &= \{\boldsymbol{b}^1, \boldsymbol{b}^2, \ldots, \boldsymbol{b}^m\}, \\
\Gamma &= \left\{ \boldsymbol{\alpha} \in \mathbb{R}^m \;\middle|\; \sum_{i=1}^{m} \alpha_i = 1,\ \alpha_i \geqq 0\ (i = 1, \ldots, m) \right\}, \\
\Omega &= \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \boldsymbol{x} = \sum_{i=1}^{m} \alpha_i \boldsymbol{b}^i,\ \boldsymbol{\alpha} \in \Gamma \right\},
\end{aligned}$$

respectively.

Consider the following nonlinear programming problem:

$$\text{(P)} \quad \begin{array}{ll} \text{Maximize} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in \Omega. \end{array}$$

Answer the following questions.

(i) Show that the following inequality holds for all $\boldsymbol{\alpha} \in \Gamma$.

$$h\left(\sum_{i=1}^{m} \alpha_i \boldsymbol{b}^i \right) \leqq \sum_{i=1}^{m} \alpha_i h(\boldsymbol{b}^i).$$

(ii) Show that functions g and f are convex.

(iii) Write out Karush-Kuhn-Tucker conditions of the following linear programming problem.

$$\begin{array}{ll} \text{Maximize} & \displaystyle\sum_{i=1}^{m} f(\boldsymbol{b}^i)\alpha_i \\ \text{subject to} & \displaystyle\sum_{i=1}^{m} \alpha_i = 1 \\ & \alpha_i \geqq 0 \ \ (i = 1, \ldots, m), \end{array}$$

where the decision variables are α_i $(i = 1, \ldots, m)$.

(iv) Let X^* be the set of optimal solutions of problem (P). Show that $X^* \cap \Delta \neq \emptyset$.