グラフ理論

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$\mathbb{R}_+$ を非負実数の集合とし，$N = [G, w]$ を単純連結無向グラフ $G = (V, E)$，枝重み $w : E \to \mathbb{R}_+$ からなるネットワークとする．$|V| = n \geqq 2$ とする．V の分割 $\pi = \{V_1, V_2, \ldots, V_p\}$ に対し，異なる節点集合 $V_i, V_j \in \pi$ 間をつなぐ枝の集合を $E(\pi)$ と記す．V のすべての分割の集合を Π と記す．

N に対しクラスカルのアルゴリズムを用いて求めた最小木を $T = (V, \{a_1, a_2, \ldots, a_{n-1}\})$ とする．ここで，a_i は木の枝として i 番目に選ばれた枝とする．森 $T_0 = (V, \emptyset)$，$T_i = (V, \{a_1, a_2, \ldots, a_i\})$，$i = 1, 2, \ldots, n-1$ に対し，$\pi_i \in \Pi$，$i = 0, 1, \ldots, n-1$ を森 T_i の連結成分が作る節点集合 V の分割と定める．$\pi_{n-1} = \{V\}$ である．各分割 $\pi \in \Pi$ に対する実数値 $y(\pi)$ を以下のように定める．

$y(\pi_0) = w(a_1)$,

$y(\pi_i) = w(a_{i+1}) - w(a_i),\ i = 1, 2, \ldots, n-2$,

$y(\pi) = 0,\ \forall \pi \in \Pi - \{\pi_0, \pi_1, \ldots, \pi_{n-2}\}$.

以下の問いに答えよ．

(i) N に対する最小木を求めるクラスカルのアルゴリズムを記述せよ．

(ii) 各枝 $e \in E$ に対し次を満たす添え字 $j(e) \in \{0, 1, \ldots, n-1\}$ が存在することを証明せよ．

$e \in E(\pi_i),\ \forall i \leqq j(e)$,

$e \notin E(\pi_i),\ \forall i > j(e)$.

(iii) 各 $i = 1, 2, \ldots, n-1$ に対し $w(a_i) \leqq w(e),\ \forall e \in E(\pi_{i-1})$ が成り立つことを証明せよ．

(iv) 各 $i = 1, 2, \ldots, n-1$ に対し $\displaystyle\sum_{j=0,1,\ldots,i-1} y(\pi_j) = w(a_i)$ が成り立つことを証明せよ．

(v) 各枝 $e \in E$ に対し $\displaystyle\sum_{\pi \in \Pi:\ e \in E(\pi)} y(\pi) \leqq w(e)$ が成り立つことを証明せよ．

An English Translation:

Graph Theory

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Let $\mathbb{R}_+$ be the set of non-negative reals, and $N = [G, w]$ be a network that consists of a simple connected graph $G = (V, E)$ and an edge weight $w : E \to \mathbb{R}_+$, and let $|V| = n \geqq 2$. For a partition $\pi = \{V_1, V_2, \ldots, V_p\}$ of V, let $E(\pi)$ denote the set of edges between distinct vertex subsets $V_i, V_j \in \pi$. Denote by Π the set of all partitions of V.

Let $T = (V, \{a_1, a_2, \ldots, a_{n-1}\})$ be a minimum spanning tree obtained from N by Kruskal's algorithm, where a_i is added to T as the i-th tree edge. For forests $T_0 = (V, \emptyset)$ and $T_i = (V, \{a_1, a_2, \ldots, a_i\})$, $i = 1, 2, \ldots, n-1$, let $\pi_i \in \Pi$, $i = 0, 1, \ldots, n-1$ be the partition formed by the connected components of forest T_i, where $\pi_{n-1} = \{V\}$. Choose a real value $y(\pi)$ for each partition $\pi \in \Pi$ as follows.

$y(\pi_0) = w(a_1)$,

$y(\pi_i) = w(a_{i+1}) - w(a_i)$, $i = 1, 2, \ldots, n-2$,

$y(\pi) = 0$, $\forall \pi \in \Pi - \{\pi_0, \pi_1, \ldots, \pi_{n-2}\}$.

Answer the following questions.

(i) Give a description of Kruskal's algorithm to find a minimum spanning tree of N.

(ii) Prove that each edge $e \in E$ admits an index $j(e) \in \{0, 1, \ldots, n-1\}$ which satisfies the conditions:

$e \in E(\pi_i)$, $\forall i \leqq j(e)$,

$e \notin E(\pi_i)$, $\forall i > j(e)$.

(iii) Prove that $w(a_i) \leqq w(e)$, $\forall e \in E(\pi_{i-1})$ holds for each $i = 1, 2, \ldots, n-1$.

(iv) Prove that $\displaystyle\sum_{j=0,1,\ldots,i-1} y(\pi_j) = w(a_i)$ holds for each $i = 1, 2, \ldots, n-1$.

(v) Prove that $\displaystyle\sum_{\pi \in \Pi:\ e \in E(\pi)} y(\pi) \leqq w(e)$ holds for each edge $e \in E$.