常微分方程式

6

$n \geqq 2$ を自然数とする．a, b を実数とし，A を対角成分が $a+b$，それ以外の成分が b の n 次正方行列とする:

$$A = \begin{pmatrix} a+b & b & \dots & b \\ b & a+b & \ddots & \vdots \\ \vdots & \ddots & \ddots & b \\ b & \dots & b & a+b \end{pmatrix}.$$

常微分方程式系

$$\frac{d\boldsymbol{x}}{dt} = A\boldsymbol{x}, \qquad \boldsymbol{x} \in \mathbb{R}^n$$

に対して，初期条件

$$\boldsymbol{x}(0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

を満たす解 $\boldsymbol{x}(t)$ を考える．以下の問いに答えよ．

(i) $n = 2$ のとき，$\boldsymbol{x}(t)$ を求めよ．

(ii) $n = 3$ のとき，$\boldsymbol{x}(t)$ を求めよ．

(iii) $n = 3$ のとき，$\lim_{t\to\infty} \boldsymbol{x}(t) = \boldsymbol{0}$ となるための必要十分条件を a, b で表せ．

(iv) 任意の自然数 $n \geqq 2$ に対して，$\lim_{t\to\infty} \boldsymbol{x}(t) = \boldsymbol{0}$ となるための必要十分条件を a, b, n で表せ．

An English Translation:

Ordinary Differential Equations

6

Let $n \geqq 2$ be an integer. Let a, b be real numbers, and A be the $n \times n$ matrix whose diagonal components are $a+b$ and whose other components are b:

$$A = \begin{pmatrix} a+b & b & \dots & b \\ b & a+b & \ddots & \vdots \\ \vdots & \ddots & \ddots & b \\ b & \dots & b & a+b \end{pmatrix}.$$

Consider the solution $\boldsymbol{x}(t)$ of the system of ordinary differential equations

$$\frac{d\boldsymbol{x}}{dt} = A\boldsymbol{x}, \qquad \boldsymbol{x} \in \mathbb{R}^n,$$

satisfying the initial condition

$$\boldsymbol{x}(0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Answer the following questions.

(i) For $n = 2$, find $\boldsymbol{x}(t)$.

(ii) For $n = 3$, find $\boldsymbol{x}(t)$.

(iii) For $n = 3$, obtain a necessary and sufficient condition for $\lim_{t\to\infty} \boldsymbol{x}(t) = \boldsymbol{0}$, and express the condition using a and b.

(iv) For any integer $n \geqq 2$, obtain a necessary and sufficient condition for $\lim_{t\to\infty} \boldsymbol{x}(t) = \boldsymbol{0}$, and express the condition using a, b and n.