凸最適化

3

$f: \mathbb{R}^n \to \mathbb{R}$ を連続的微分可能な凸関数とする．さらに，$\boldsymbol{A}$ を $m \times n$ の行列とし，$\boldsymbol{b}$ を m 次元ベクトルとする．

次の凸最適化問題 (P) を考える．

$$
\begin{array}{lll}
\text{(P)} & \text{Minimize} & f(\boldsymbol{x}) \\
 & \text{subject to} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\
 & & \boldsymbol{x} \geqq \boldsymbol{0}
\end{array}
$$

ただし，決定変数は $\boldsymbol{x} \in \mathbb{R}^n$ である．問題 (P) は最適解をもつと仮定する．さらに，X^* を問題 (P) の最適解の集合とする．

以下の問いに答えよ．

(i) 任意の $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ に対して，次の不等式が成り立つことを示せ．

$$
(\nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y}))^\top (\boldsymbol{x} - \boldsymbol{y}) \geqq 0
$$

(ii) 問題 (P) のカルーシュ・キューン・タッカー (Karush-Kuhn-Tucker) 条件を書け．

(iii) $\boldsymbol{x}^* \in X^*$ とする．次の線形計画問題 (Q) を考える．

$$
\begin{array}{lll}
\text{(Q)} & \text{Minimize} & \nabla f(\boldsymbol{x}^*)^\top \boldsymbol{y} \\
 & \text{subject to} & \boldsymbol{A}\boldsymbol{y} = \boldsymbol{b} \\
 & & \boldsymbol{y} \geqq \boldsymbol{0}
\end{array}
$$

ただし，決定変数は $\boldsymbol{y} \in \mathbb{R}^n$ である．問題 (Q) の双対問題を書け．さらに，問題 (Q) が最適解を持つことを示せ．

(iv) 任意の $\boldsymbol{x}^*, \boldsymbol{y}^* \in X^*$ に対して，以下の式が成り立つことを示せ．

$$
(\nabla f(\boldsymbol{x}^*) - \nabla f(\boldsymbol{y}^*))^\top (\boldsymbol{x}^* - \boldsymbol{y}^*) = 0
$$

An English Translation:

Convex Optimization

$\boxed{3}$

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable convex function. Moreover, let $\boldsymbol{A}$ and $\boldsymbol{b}$ be an $m \times n$ matrix and an m-dimensional vector, respectively.

Consider the following convex optimization problem (P):

$$\begin{array}{lll} \text{(P)} & \text{Minimize} & f(\boldsymbol{x}) \\ & \text{subject to} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & & \boldsymbol{x} \geqq \boldsymbol{0}, \end{array}$$

where the decision variable is $\boldsymbol{x} \in \mathbb{R}^n$. Suppose that problem (P) has an optimal solution. Let X^* be the set of optimal solutions of problem (P).

Answer the following questions.

(i) Show that the following inequality holds for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.

$$(\nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y}))^\top (\boldsymbol{x} - \boldsymbol{y}) \geqq 0.$$

(ii) Write out Karush-Kuhn-Tucker conditions of problem (P).

(iii) Let $\boldsymbol{x}^* \in X^*$. Consider the following linear programming problem (Q):

$$\begin{array}{lll} \text{(Q)} & \text{Minimize} & \nabla f(\boldsymbol{x}^*)^\top \boldsymbol{y} \\ & \text{subject to} & \boldsymbol{A}\boldsymbol{y} = \boldsymbol{b} \\ & & \boldsymbol{y} \geqq \boldsymbol{0}, \end{array}$$

where the decision variable is $\boldsymbol{y} \in \mathbb{R}^n$. Write out a dual problem of problem (Q). Moreover, show that problem (Q) has an optimal solution.

(iv) Show that the following equation holds for any $\boldsymbol{x}^*, \boldsymbol{y}^* \in X^*$:

$$(\nabla f(\boldsymbol{x}^*) - \nabla f(\boldsymbol{y}^*))^\top (\boldsymbol{x}^* - \boldsymbol{y}^*) = 0.$$