凸最適化

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$\boldsymbol{A} \in \mathbb{R}^{m\times n}$，$\boldsymbol{b} \in \mathbb{R}^{m}$，$\boldsymbol{C} \in \mathbb{R}^{n\times n}$ とする．パラメータ $\boldsymbol{x} = (x_1, \ldots, x_n)^\top \in \mathbb{R}^n$ をもつ次の非線形計画問題を考える．

$$\begin{aligned} \mathrm{P}(\boldsymbol{x}): \quad & \text{Minimize} \quad \sum_{i=1}^{n} (\boldsymbol{z}^i)^\top \boldsymbol{z}^i + \boldsymbol{y}^\top \boldsymbol{y} + \boldsymbol{x}^\top \boldsymbol{C} \boldsymbol{x} \\ & \text{subject to} \quad \boldsymbol{y} - \sum_{i=1}^{n} x_i \boldsymbol{z}^i = \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b} \end{aligned}$$

ここで，$\mathrm{P}(\boldsymbol{x})$ の決定変数は $\boldsymbol{y}, \boldsymbol{z}^i \in \mathbb{R}^m$ $(i = 1, \ldots, n)$ である．また，$^\top$ は転置記号を表す．さらに，任意の $\boldsymbol{x}$ に対して，問題 $\mathrm{P}(\boldsymbol{x})$ の最適値が定義されているとし，その最適値を $f(\boldsymbol{x})$ と表す．

以下の問いに答えよ．

(i) 問題 $\mathrm{P}(\boldsymbol{x})$ のカルーシュ・キューン・タッカー条件 (Karush-Kuhn-Tucker 条件) を書け．

(ii) 問題 $\mathrm{P}(\boldsymbol{x})$ の目的関数が，$\boldsymbol{y}, \boldsymbol{z}^i \in \mathbb{R}^m$ $(i = 1, \ldots, n)$ に対して凸であることを示せ．

(iii) $\boldsymbol{C}$ を正定値対称行列と仮定し，次の最適化問題を考える．

$$\begin{aligned} \mathrm{P1}: \quad & \text{Minimize} \quad f(\boldsymbol{x}) \\ & \text{subject to} \quad \boldsymbol{x} \in \mathbb{R}^n \end{aligned}$$

$\boldsymbol{x}^* \in \mathbb{R}^n$ を問題 P1 の大域的最適解とするとき，以下の不等式が成り立つことを示せ．

$$(\boldsymbol{x}^*)^\top \boldsymbol{x}^* \leqq \frac{\boldsymbol{b}^\top \boldsymbol{b}}{\lambda_{\min}(\boldsymbol{C})}$$

ただし，$\lambda_{\min}(\boldsymbol{C})$ は $\boldsymbol{C}$ の最小固有値を表す．

(iv) $\boldsymbol{A}$ を $m \times n$ 零行列，$\boldsymbol{b}$ を m 次元零ベクトルと仮定する．以下の最適化問題を考える．

$$\begin{aligned} \mathrm{P2}: \quad & \text{Minimize} \quad f(\boldsymbol{x}) \\ & \text{subject to} \quad \boldsymbol{x}^\top \boldsymbol{x} \leqq \alpha \end{aligned}$$

ここで，$\alpha \in \mathbb{R}$ は正の定数である．$(\hat{\boldsymbol{x}}, \rho), (\bar{\boldsymbol{x}}, \rho) \in \mathbb{R}^n \times \mathbb{R}$ が共に問題 P2 のカルーシュ・キューン・タッカー条件を満たすとき，$f(\hat{\boldsymbol{x}}) = f(\bar{\boldsymbol{x}})$ が成り立つことを示せ．

An English Translation:

Convex Optimization

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Let $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{b} \in \mathbb{R}^m$ and $\boldsymbol{C} \in \mathbb{R}^{n\times n}$. Consider the following nonlinear programming problem with parameter $\boldsymbol{x} = (x_1, \ldots, x_n)^\top \in \mathbb{R}^n$:

$$\begin{aligned} \mathrm{P}(\boldsymbol{x}): \quad & \text{Minimize} \quad \sum_{i=1}^{n} (\boldsymbol{z}^i)^\top \boldsymbol{z}^i + \boldsymbol{y}^\top \boldsymbol{y} + \boldsymbol{x}^\top \boldsymbol{C} \boldsymbol{x} \\ & \text{subject to} \quad \boldsymbol{y} - \sum_{i=1}^{n} x_i \boldsymbol{z}^i = \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}, \end{aligned}$$

where the decision variables of $\mathrm{P}(\boldsymbol{x})$ are $\boldsymbol{y}, \boldsymbol{z}^i \in \mathbb{R}^m$ $(i = 1, \ldots, n)$, with $^\top$ denoting transposition. Moreover, denote by $f(\boldsymbol{x})$ the optimal value of problem $\mathrm{P}(\boldsymbol{x})$, assuming that it is well-defined for all $\boldsymbol{x}$.

Answer the following questions.

(i) Write out the Karush-Kuhn-Tucker conditions of $\mathrm{P}(\boldsymbol{x})$.

(ii) Prove that the objective function of problem $\mathrm{P}(\boldsymbol{x})$ is convex with respect to $\boldsymbol{y}, \boldsymbol{z}^i \in \mathbb{R}^m$ $(i = 1, \ldots, n)$.

(iii) Assume that $\boldsymbol{C}$ is symmetric positive definite and consider the following optimization problem:

$$\begin{aligned} \mathrm{P1}: \quad & \text{Minimize} \quad f(\boldsymbol{x}) \\ & \text{subject to} \quad \boldsymbol{x} \in \mathbb{R}^n. \end{aligned}$$

Show that the following inequality holds when $\boldsymbol{x}^* \in \mathbb{R}^n$ is a global optimal solution of problem P1:

$$(\boldsymbol{x}^*)^\top \boldsymbol{x}^* \leqq \frac{\boldsymbol{b}^\top \boldsymbol{b}}{\lambda_{\min}(\boldsymbol{C})},$$

where $\lambda_{\min}(\boldsymbol{C})$ denotes the smallest eigenvalue of $\boldsymbol{C}$.

(iv) Assume that $\boldsymbol{A}$ is the $m \times n$ zero matrix and $\boldsymbol{b}$ is the m-dimensional zero vector. Consider the following optimization problem:

$$\begin{aligned} \mathrm{P2}: \quad & \text{Minimize} \quad f(\boldsymbol{x}) \\ & \text{subject to} \quad \boldsymbol{x}^\top \boldsymbol{x} \leqq \alpha, \end{aligned}$$

where $\alpha \in \mathbb{R}$ is a positive constant. Show that $f(\hat{\boldsymbol{x}}) = f(\bar{\boldsymbol{x}})$ holds, when both $(\hat{\boldsymbol{x}}, \rho), (\bar{\boldsymbol{x}}, \rho) \in \mathbb{R}^n \times \mathbb{R}$ satisfy the Karush-Kuhn-Tucker conditions of problem P2.